

Different Forms of the Governing Equations for Atmospheric Motions

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Outline

- **Conservation relations and approximate equations for motion on the sphere**
 - **Nonhydrostatic “deep” equations**
 - **Hydrostatic Primitive equations**
- **Filtering acoustic waves in nonhydrostatic models**
 - **Boussinesq approximation**
 - **Anelastic and Pseudo-incompressible systems**
 - **Application to global models**

Approximating the Equations on the Sphere

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Consistent approximate models of the global atmosphere: shallow, deep, hydrostatic, quasi-hydrostatic and non-hydrostatic

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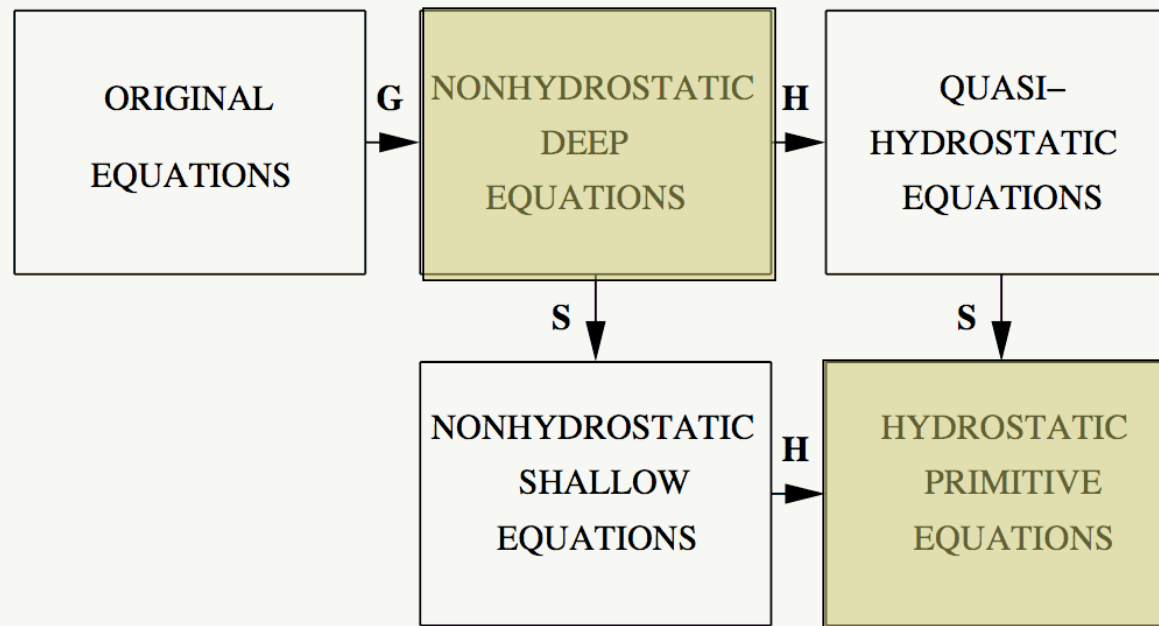


Figure 4. Showing the interrelationships of the four consistent approximate models of the global atmosphere identified in this study (NHD, QHE, NHS and HPE models) and the relationship of the NHD model to the original (unapproximated) equations. **G** denotes the spherical geopotential approximation, **H** the omission of the term Dw/Dt from the vertical component of the momentum equation, and **S** the shallow atmosphere combination of approximations (see text).

Spherical Coordinate System

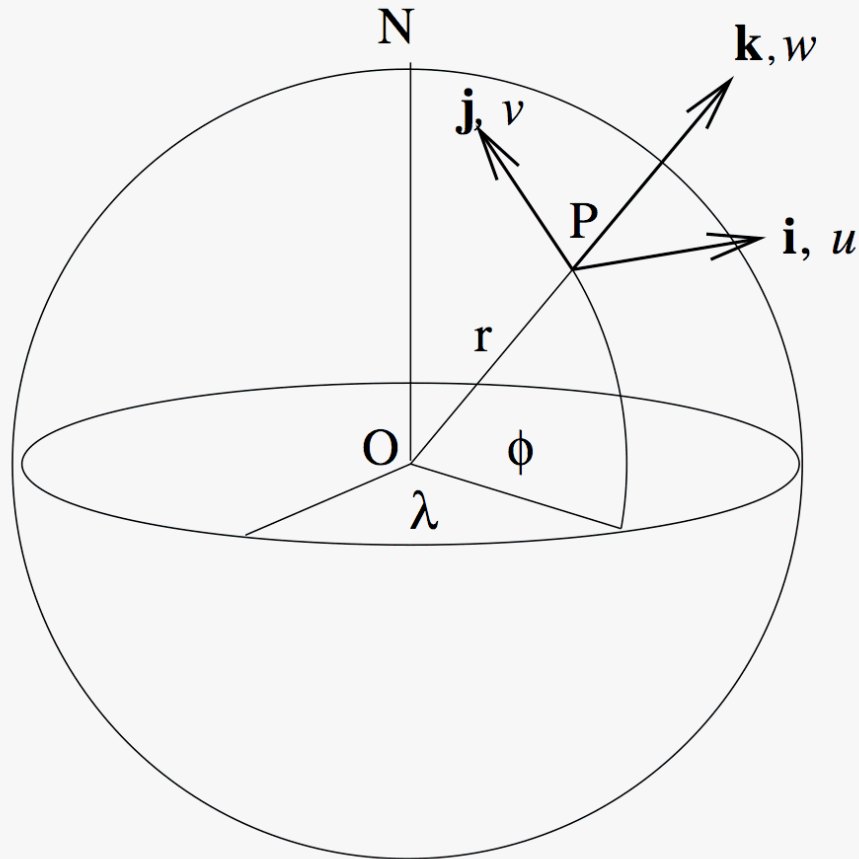


Figure 2. The λ, ϕ, r spherical polar coordinate system. λ is longitude, ϕ latitude of point P and r its distance from the origin O . The polar axis ON ($\phi = \pi/2$) coincides with the rotation axis of the Earth. Arrows at P indicate the local unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ associated with the corresponding λ, ϕ, r directions; velocity components u, v, w are also shown.

Nonhydrostatic Deep Momentum Equations

$$\begin{aligned}\frac{Du}{Dt} &= -\frac{uw}{r} + \frac{uv}{r} \tan \phi + 2\Omega v \sin \phi - 2\Omega w \cos \phi - \frac{\alpha}{r \cos \phi} \frac{\partial p}{\partial \lambda} + G_\lambda; \\ \frac{Dv}{Dt} &= -\frac{vw}{r} - \frac{u^2}{r} \tan \phi - 2\Omega u \sin \phi - \frac{\alpha}{r} \frac{\partial p}{\partial \phi} + G_\phi; \\ \frac{Dw}{Dt} &= \frac{u^2}{r} + \frac{v^2}{r} + 2\Omega u \cos \phi - g - \alpha \frac{\partial p}{\partial r} + G_r.\end{aligned}$$

Material derivative is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\lambda} \frac{\partial}{\partial \lambda} + \dot{\phi} \frac{\partial}{\partial \phi} + \dot{r} \frac{\partial}{\partial r} = \frac{\partial}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial r}.$$

- r is the distance to the center of the earth
- All Coriolis terms retained
- All metric terms retained

Conservation Relations for NDE

Axial angular momentum

$$(u + \Omega r \cos \phi)r \cos \phi$$

Ertel potential vorticity

$$\mathbf{Z} = 2\boldsymbol{\Omega} + \nabla \times \mathbf{u}$$
$$\alpha \mathbf{Z} \cdot \nabla \theta$$

Total energy per unit mass

$$\frac{1}{2} \mathbf{u}^2 + \Phi_A(r) + c_v T$$

Hydrostatic Primitive Equations

$$\frac{D_a u}{Dt} = \frac{uv}{a} \tan \phi + 2\Omega v \sin \phi - \frac{\alpha}{a \cos \phi} \frac{\partial p}{\partial \lambda} + G_\lambda,$$

$$\frac{D_a v}{Dt} = -\frac{u^2}{a} \tan \phi - 2\Omega u \sin \phi - \frac{\alpha}{a} \frac{\partial p}{\partial \phi} + G_\phi,$$

$$g + \alpha \frac{\partial p}{\partial z} = 0,$$

Material derivative is:

$$\frac{D_a}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_a$$

- hydrostatic approximation
- z replaces r as the vertical coordinate
- a replaces r as the distance to the center of the earth
- $\cos \phi$ Coriolis terms neglected
- only two of six metric terms retained

Conservation Relations for HPE

Axial angular momentum

$$(u + \Omega a \cos \phi)a \cos \phi$$

Potential vorticity $\mathbf{v} \equiv (u, v, 0)$

$$\mathbf{Z}_{\text{HPE}} = f\mathbf{k} + \nabla_a \times \mathbf{v}$$
$$\alpha \mathbf{Z}_{\text{HPE}} \cdot \nabla_a \theta$$

Total energy per unit mass

$$\frac{1}{2}\mathbf{v}^2 + gz + c_v T$$

Neglect of $\cos\phi$ Coriolis terms

- Required to obtain consistent conservation relations in the primitive equations.
- “Subject of quiet controversy in meteorology and oceanography for many years.” (White, et al., 2005)
 - Not important according to linearized adiabatic analyses.
 - Various scaling analyses suggest they may nevertheless be non-negligible, particularly in the tropics.

Simple Argument for Including the $\cos\phi$ Coriolis Terms

- Their neglect introduces a false variation of the Coriolis force with latitude.
- There is *no* spatial dependence in the Coriolis force.
 - Coriolis force on velocity vector \mathbf{u} is identical at the pole and the equator.
 - Compare the position and velocity dependence

$$-2\boldsymbol{\Omega} \times \mathbf{u}$$

$$-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

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Generalizing the Boussinesq Approximation to Stratified Compressible Flow



Joseph Valentin Boussinesq (1842-1929)

The Approximation of Incompressibility

- Assume fluid is *incompressible* (density is independent of pressure)
 - Adiabatic flow
 - Neglect diffusion of heat through sides of a fluid parcel

$$\frac{D\rho}{Dt} = 0$$

Boussinesq Continuity Relation

- Having made the incompressibility approximation

$$\frac{D\rho}{Dt} = 0$$

- Mass continuity

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0$$

- Reduces to volume conservation

$$\nabla \cdot \mathbf{u} = 0$$

The Boussinesq System

(inviscid, adiabatic)

$$\frac{D\mathbf{u}}{Dt} + \nabla P = b\mathbf{k}, \quad \frac{Db}{Dt} + N^2 w = 0, \quad \nabla \cdot \mathbf{u} = 0$$

- P : pressure potential
- b : buoyancy
- N^2 : Brunt Vaisala frequency

P is computed from a diagnostic, elliptic PDE
Acoustic waves are eliminated

Defining b , P and N : Incompressible Fluids

- Split ρ into a vertically varying component and the remainder

$$\rho(x, y, z, t) = \bar{\rho}(z) + \rho'(x, y, z, t)'$$

- Define

$$b = -g \frac{\rho'}{\rho_0}, \quad N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz},$$

- Then

$$\frac{D\rho}{Dt} = 0 \rightarrow \frac{Db}{Dt} + N^2 w = 0$$

Defining b , P and N : Incompressible Fluids

- Split p like ρ such that the vertically varying components are in hydrostatic balance. Without approximation

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla p = -g \frac{\rho'}{\rho} \mathbf{k}$$

- Define

$$P = \frac{p'}{\rho_0}$$

- Then the approximate momentum equation is

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho_0} \nabla p = -g \frac{\rho'}{\rho_0} \mathbf{k}$$

Boussinesq Approximation for Compressible Fluids - I

- Assume ideal gas, adiabatic flow
 - Thermodynamic equation is

$$\frac{D\theta}{Dt} = 0$$

- Also note

$$\frac{1}{\rho} \nabla p = c_p \theta \nabla \pi, \quad \pi = \left(\frac{p}{p_s} \right)^{R/c_p}$$

Boussinesq Approximation for Compressible Fluids - II

- Split θ into a vertically varying component and the remainder

$$\theta(x, y, z, t) = \bar{\theta}(z) + \theta'(x, y, z, t)$$

- Define

$$b = g \frac{\theta'}{\theta_0}, \quad N^2 = \frac{g}{\theta_0} \frac{d\bar{\theta}}{dz}$$

- Then

$$\frac{D\theta}{Dt} = 0 \rightarrow \frac{Db}{Dt} + N^2 w = 0$$

Boussinesq Approximation for Compressible Fluids - III

- Split π like θ such that the vertically varying components are in hydrostatic balance. Without approximation

$$\frac{D\mathbf{u}}{Dt} + c_p \theta \nabla \pi' = g \frac{\theta'}{\theta} \mathbf{k}$$

- Define

$$P = c_p \theta_0 \pi'$$

- Then the approximate momentum equation is

$$\frac{D\mathbf{u}}{Dt} + c_p \theta_0 \nabla \pi' = g \frac{\theta'}{\theta_0} \mathbf{k}$$

Boussinesq Approximation for Compressible Fluids - IV

- We have not assumed

$$\frac{D\rho}{Dt} = 0$$

- So reducing mass continuity

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0$$

- to

$$\nabla \cdot \mathbf{u} = 0$$

- is crude.

Anelastic Models - I

- For stratified compressible flow replace the non-divergence condition with the anelastic continuity equation

$$\frac{w}{\bar{\rho}} \frac{d\bar{\rho}}{dz} + \nabla \cdot \mathbf{u} = 0$$

Anelastic Models - II

- Reference profile is adiabatic
 - Ogura and Phillips (1962)
 - Energy conservative
- Reference profile is best match for actual environment
 - Wilhelmson and Ogura (1972)
 - Not energy conservative
 - Lipps and Hemler (1982)
 - Approximate **momentum** equations & recover conservation
 - Bannon (1996)
 - Approximate **thermodynamic** equation & recover conservation

Pseudo-Incompressible Approximation

- Define pseudo-density ρ^* and enforce mass conservation with respect to ρ^* (Durran, 1989)

$$\frac{1}{\rho^*} \frac{D\rho^*}{Dt} + \nabla \cdot \mathbf{u} = 0$$

- Sound waves are eliminated if $\rho^* = f(\text{entropy}, \mathbf{x}, t)$
- For perfect gases, choose

$$\rho^* = \bar{\rho} \bar{\theta} / \theta$$

- If the flow is isentropic, changes in p have no influence on ρ^*

(ρ^* behaves as if the fluid were incompressible)

Alternate Form

- Can use the thermodynamic equation to eliminate $d\theta/dt$

$$\nabla \cdot (\bar{\rho}\bar{\theta}\mathbf{u}) = \frac{\bar{\rho}Q}{c_p\bar{\pi}}$$

- Q is heating rate per unit mass
- May be combined with the **unapproximated** momentum and thermodynamic equations to yield an energy conservative system.

Lagrangian Energy Relations - 1

- Suppose the deviations of the **thermodynamic** variables from their reference state values are small

$$\rho' / \bar{\rho}, \quad p' / \bar{p}, \quad \theta' / \bar{\theta}, \quad \pi' / \bar{\pi} \ll 1$$

- KE equation becomes

$$\bar{\rho} \frac{D}{Dt} \left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) = -\nabla \cdot (p' \mathbf{u}) + \frac{p'}{\bar{\rho}} \left[\nabla \cdot (\bar{\rho} \mathbf{u}) + \bar{\rho} w \frac{d \ln \bar{\theta}}{dz} \right] + \bar{\rho} g w \frac{\theta'}{\bar{\theta}}$$

Lagrangian Energy Relations - 2

- Potential energy gz and anelastic enthalpy

$$(c_p \pi \theta = c_p T = c_v T + p/\rho)$$

$$\bar{\rho} \frac{D}{Dt} (c_p \bar{\pi} \theta + gz) = -\bar{\rho} g w \frac{\theta'}{\bar{\theta}} + \bar{\rho} Q$$

- Elastic internal energy

$$\bar{\rho} \frac{D}{Dt} \left[\frac{1}{2c_s^2} \left(\frac{p'}{\bar{\rho}} \right)^2 \right] = -\frac{p'}{\bar{\rho}} \left[\nabla \cdot (\bar{\rho} \mathbf{u}) + \bar{\rho} w \frac{d \ln \bar{\theta}}{dz} \right] + \frac{p' Q}{c_p \bar{T}}$$

Fundamental Anelastic Approximation

- Elastic energy is either negligible or in equilibrium
- Thus

$$\nabla \cdot (\bar{\rho} \mathbf{u}) + \bar{\rho} w \frac{d \ln \bar{\theta}}{dz} = \frac{\bar{\rho} Q}{c_p \bar{T}}$$

- Identical to the pseudo-incompressible continuity equation!

Other Anelastic Systems

- Continuity equation is $\nabla \cdot (\bar{\rho}\mathbf{u}) = 0$
- Need

$$\nabla \cdot (\bar{\rho}\mathbf{u}) + \bar{\rho}w \frac{d \ln \bar{\theta}}{dz} = 0$$

- Energy conservation is achieved by
 - Using an isentropic reference state
 - Changing other governing equations to eliminate the term in red from the KE equation forcing.

Comparing Sound-Proof Systems

TABLE 3. HEIGHT-SCALE DISTORTION, ENERGY REDISTRIBUTION, AND RELOCATION OF ZEROS (SEE TEXT FOR DEFINITION OF THESE) OF INTERNAL MODES AS A FUNCTION OF EQUATION SET

Equation set	Height-scale distortion	Energy redistribution		Relocation of zeros
		Rossby	Gravity	
Fully compressible	No	No	No	No
Hydrostatic	No	No	Yes	No
Pseudo-incompressible (Durrant 1989)	No	No	Yes	No
Anelastic (Wilhelmson and Ogura 1972)	Yes	Yes	Yes	Yes
Anelastic (Lipps and Hemler 1982)	No	Yes	Yes	Yes
Boussinesq	Yes	Yes	Yes	Yes

TERRY DAVIES, ANDREW STANIFORTH, NIGELWOOD and JOHN THUBURN: Validity of anelastic and other equation sets as inferred from normal-mode analysis. *Q. J. R. Meteorol. Soc.* (2003), **129**, 2761–2775

Energy Redistribution

- Modal structure (isothermal atmosphere)

$$(u_0, v_0, w_0, \theta_0, \pi_0) e^{i(kx + mz - \omega t)}$$

- Change relative magnitudes of u, v, w, θ, π .

$$u_0 = \frac{\omega k c_p \theta_0}{\omega^2 - f^2} \pi_0$$

- Produced by errors in the frequency.

Dispersion Relations, $f = 0$

- Approximate compressible

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2 + \Gamma^2 + N^2/c_s^2}$$

- Pseudo-incompressible

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2 + \Gamma^2}$$

When are the errors large?

- For long horizontal and vertical wavelengths

$$\frac{\Gamma^2}{N^2/c_s^2} = \frac{9}{40}$$

- For $T = 250$ K isothermal atmosphere

$$\frac{2\pi}{N/c_s} \approx 100 \text{ km}$$

Normal Mode Analysis - 1

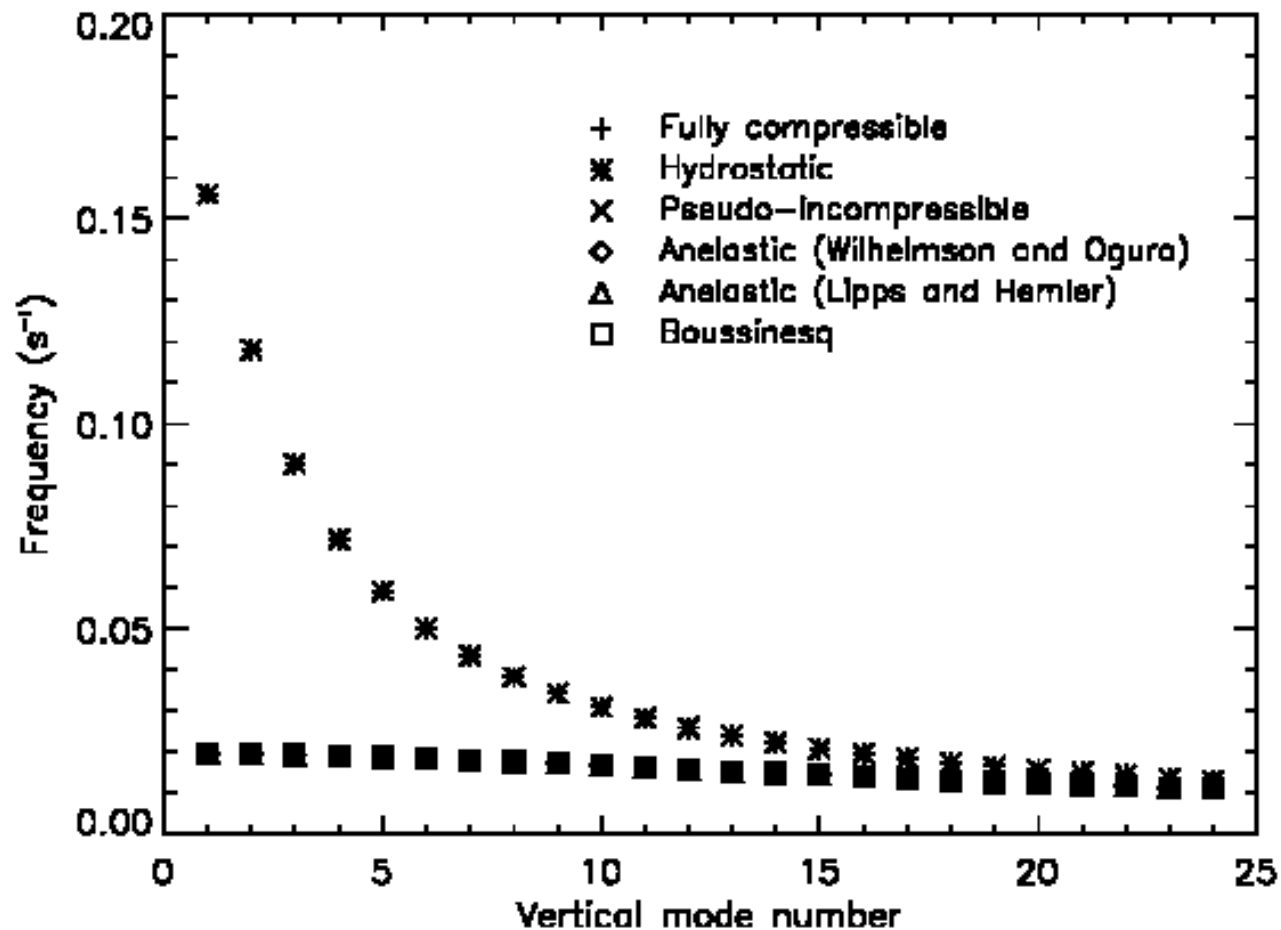


Figure 1. Frequency σ vs. vertical internal mode number m , where $k_z \equiv m\pi/z_T$ is vertical wave number, for the equation sets considered herein. Results are for a rigid lid at $z_T = 80$ km and a horizontal wavelength of $2\pi/k_x = 10$ km.

Normal Mode Analysis - 2

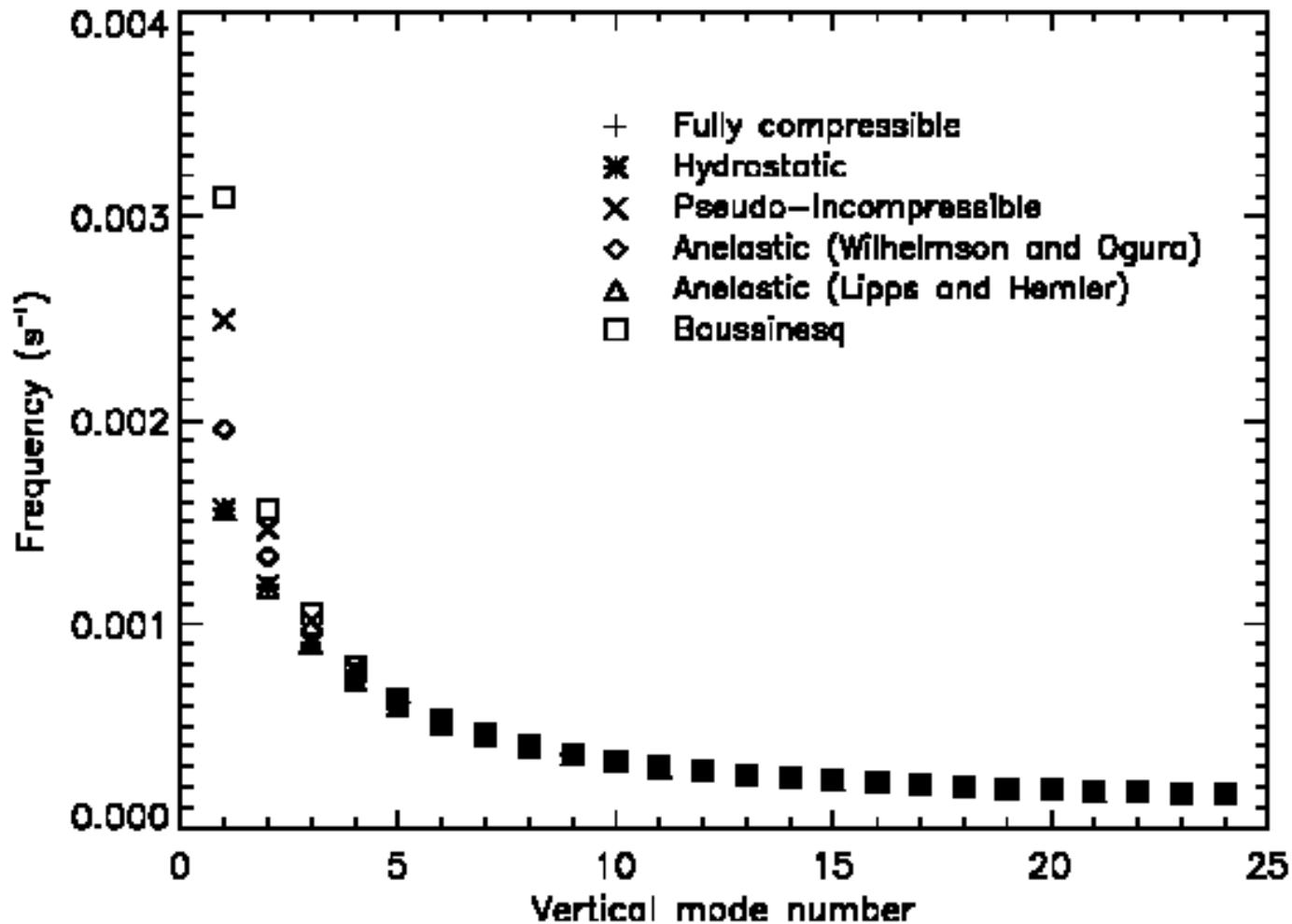


Figure 2. Same as for Fig. 1, except for a horizontal wavelength of 1000 km.

Normal Mode Analysis - 3

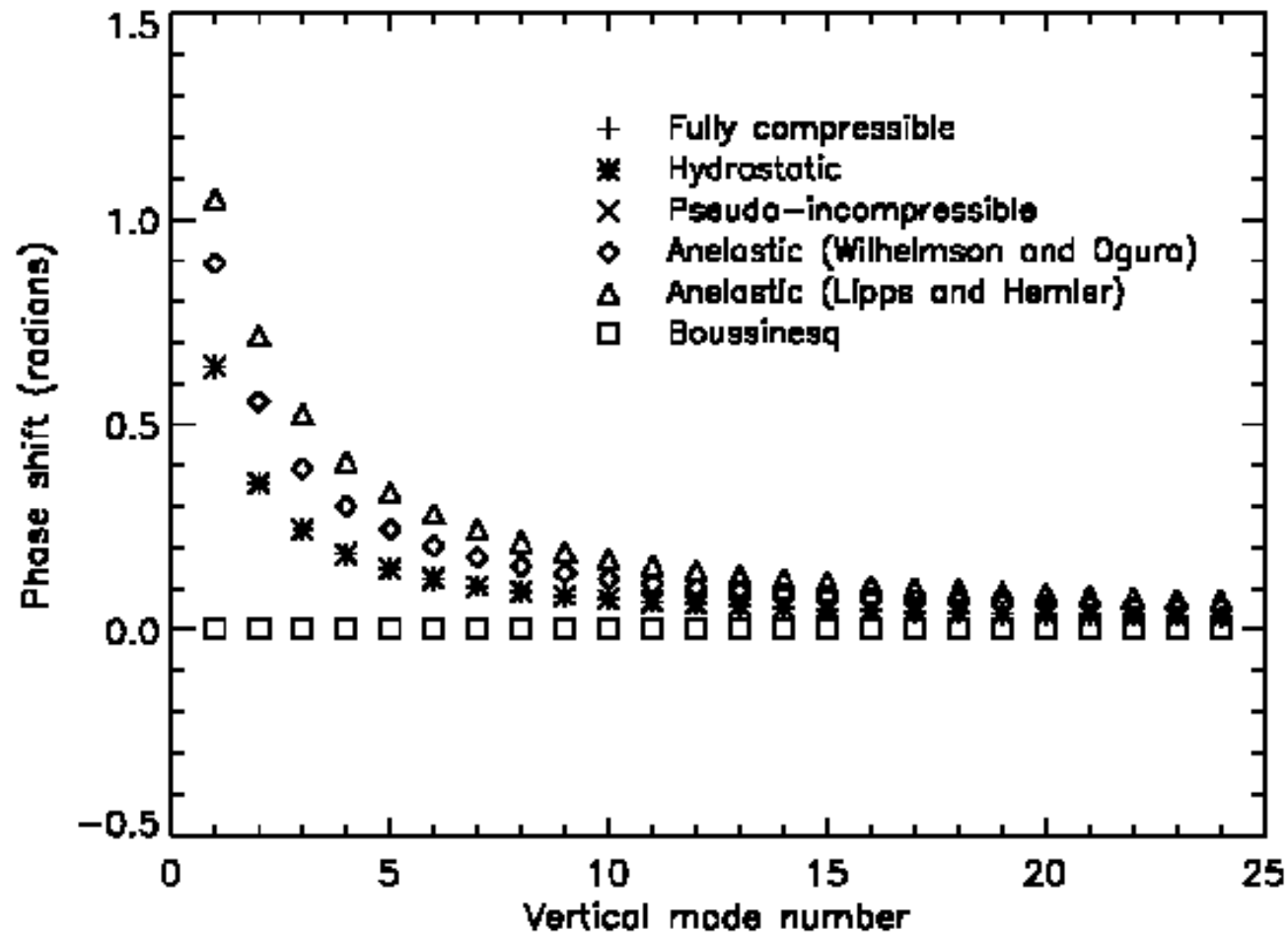


Figure 3. Phase shift Δ vs. vertical internal mode number m , where $k_z \equiv m\pi/z_T$ is vertical wave number, for the equation sets considered herein. Results are for a rigid lid at $z_T = 80$ km and are independent of horizontal wavelength $2\pi/k_x$. Note that the fully-compressible, hydrostatic and pseudo-incompressible points all coincide.

Generalized Pseudo-Incompressible System

Compute the reference state using a coarse resolution hydrostatic model (e.g., a global PE model)

$$\rho^* = \frac{\tilde{\rho}(x, y, z, t)\tilde{\theta}(x, y, z, t)}{\theta}.$$

For scaling analysis, break the reference pressure into a horizontally uniform part and the remainder.

$$\tilde{\pi} = \tilde{\pi}_v(z, t) + \tilde{\pi}_h(x, y, z, t), \quad |\tilde{\pi}_h| \ll |\tilde{\pi}_v|$$

Generalized Pseudo-Incompressible System (f -plane case)

$$\frac{D\mathbf{u}_h}{Dt} + f\mathbf{k} \times \mathbf{u}_h + c_p\theta\nabla_h(\tilde{\pi}_h + \pi') = 0$$

$$\frac{Dw}{Dt} + c_p\theta\frac{\partial\pi'}{\partial z} = g\frac{\theta'}{\tilde{\theta}}$$

$$\frac{D\theta}{Dt} = \frac{Q}{c_p\tilde{\pi}}$$

$$\frac{\partial\tilde{\rho}\tilde{\theta}}{\partial t} + \nabla \cdot (\tilde{\rho}\tilde{\theta}\mathbf{u}) = \frac{\tilde{\rho}Q}{c_p\tilde{\pi}}$$

Conditions for the validity of the generalized pseudo-incompressible system

Scale for $\nabla_h \tilde{\pi}_h \leq$ scale for $\nabla_h \pi'$ and $\mathcal{M}^2 \ll \min(1, \mathcal{R}^2)$

- **Mesoscale:** Mach number less than 1.
- **Synoptic scale:** Mach number less than the Rossby number (just satisfied),
- **Global scale:** not satisfied

Scale for $\nabla_h \tilde{\pi}_h \gg$ scale for $\nabla_h \pi'$

- **Mesoscale:** not satisfied
- **Synoptic and global scales:** hydrostatic reference pressure perturbations dominate pseudo-incompressible contribution (easily satisfied)

Eulerian Energy Conservation - 1

- Fully nonlinear
- Define the mechanical energy per unit mass

$$M = \frac{\mathbf{u} \cdot \mathbf{u}}{2} + gz$$

Eulerian Energy Conservation - 2

Pseudo-incompressible

$$\frac{\pi'}{\tilde{\pi}} \frac{\partial}{\partial t} (\tilde{\rho} c_v \tilde{T}) + \frac{\partial}{\partial t} (\rho^* M + c_v \tilde{\rho} \tilde{T}) + \nabla \cdot \left[\rho^* \left(M + c_v T + \frac{p}{\rho} \right) \mathbf{u} \right] = 0$$

Fully compressible

$$\frac{\partial}{\partial t} (\rho M + c_v \rho T) + \nabla \cdot \left[\rho \left(M + c_v T + \frac{p}{\rho} \right) \mathbf{u} \right] = 0$$

Pseudo-Incompressible Ertel PV

$$\frac{D}{Dt} \left(\frac{\boldsymbol{\omega} \cdot \nabla \theta}{\rho^*} \right) = 0$$

Elliptic Pressure Equation

$$c_p \nabla \cdot (\tilde{\rho} \tilde{\theta} \nabla \pi') = -\nabla \cdot (\tilde{\rho} \tilde{\theta} \mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{k} \times \nabla_h (\tilde{\rho} \tilde{\theta} \mathbf{u}_h) \\ + g \frac{\partial \tilde{\rho} \tilde{\theta}'}{\partial z} - c_p \nabla_h \cdot (\tilde{\rho} \tilde{\theta} \nabla_h \tilde{\pi}_h) - \frac{\partial}{\partial t} \left(\frac{\tilde{\rho} Q}{c_p \tilde{\pi}} \right) + \frac{\partial^2 \tilde{\rho} \tilde{\theta}}{\partial t^2}.$$

- Time derivative term is diagnostic
- Constant may be added to π' to ensure exact domain-averaged energy conservation
- Actual solution via projection method

Conclusions - 1

- The nonhydrostatic deep equations are an attractive improvement to the hydrostatic primitive equations
 - Nonhydrostatic
 - More faithful conservation of axial angular momentum and PV.
 - Full Coriolis force

Conclusions - 2

- Acoustic waves may be filtered by enforcing mass conservation with respect to a pseudo-density of the form $\rho^* = f(\text{entropy}, \mathbf{x}, t)$

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot (\rho^* \mathbf{u}) = 0$$

- If the motion is isentropic, the pseudo-density behaves as if the fluid were incompressible.
 - At constant entropy, ρ^* is independent of pressure
 - “Isentropic incompressibility”

Conclusions - 3

- Attractive energy conservation properties and accuracy when

$$\rho^* = \frac{\tilde{\rho}(x, y, z, t)\tilde{\theta}(x, y, z, t)}{\theta}$$

- Possible foundation for global “anelastic” modeling.
 - Reference-state thermodynamic fields computed by coarse resolution PE model.

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Relation to Pseudo-Height Coordinates - 1

- Transform continuity equation to new vertical coordinate ζ

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial \zeta} \right) + \nabla_{\zeta} \cdot \left(\rho \mathbf{u}_h \frac{\partial z}{\partial \zeta} \right) + \frac{\partial}{\partial \zeta} \left(\rho \dot{\zeta} \frac{\partial z}{\partial \zeta} \right) = 0$$

- Let ζ be pseudo-height (Hoskins and Bretherton 1972)

$$\zeta = \frac{c_p \theta_0}{g} (1 - \pi) \qquad \frac{\partial z}{\partial \zeta} = \frac{\theta}{\theta_0}.$$

Relation to Pseudo-Height Coordinates - 2

- Continuity becomes

$$\frac{\partial}{\partial t} (\rho\theta) + \nabla_{\zeta} \cdot (\rho\theta\mathbf{u}_h) + \frac{\partial}{\partial \zeta} (\rho\theta\dot{\zeta}) = 0$$

- But $\rho\theta = f(p)$, so

$$\nabla_{\zeta} \cdot (\rho\theta\mathbf{u}_h) + \frac{\partial \rho\theta\dot{\zeta}}{\partial \zeta} = 0$$

Just the continuity equation in a hydrostatic pressure coordinate.