Conservative Semi-LAgrangian Multi-tracer scheme (CSLAM): a semi-implicit shallow water and a fully compressible non-hydrostatic solver with fully consistent transport

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1 **CSLAM transport scheme**
   - formulation
   - air-tracer mass coupling
   - conditions for local mass-conservation
   - extension to there sphere: gnomonic coordinates
   - CSLAM in NCAR’s CAM-SE (Community Atmosphere Model - Spectral Element)

2 **CSLAM-SW: shallow water model**
   - semi-implicit time-stepping of tracers
   - treatment of divergence

3 **CSLAM-NH: non-hydrostatic $x - z$ plane solver**
   - preliminary results
Multi-tracer efficiency

→ CAM5 has 31 continuity equations
  (micro-physics and convection scheme developers are eager to add more!)

→ CAM-Chem has approximately 107 continuity equations

• Shape-preservation (large gradients, physics)
• Consistency (air ↔ tracer)
• Correlation accuracy
• Efficiency on ‘traditional\(^1\)’ massively parallel machines:
  • minimize frequency of message passing: e.g. long $\Delta t$’s (semi-Lagrangian)
  • minimize message sizes: local computational stencil
  • minimize memory usage: e.g., 2-time-level, no multi-moment
• Accuracy on non-orthogonal grids (splitting errors) ⇒ fully two-dimensional methods

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\(^1\)GPUs and Intel MIC architectures??
Design (see fuller discussion in Lauritzen et al., 2011)

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Specific humidity, cloud liquid water and ice: \( 10^{-4} \text{ kg/kg} \)

Very ‘oscillatory’ fields:
Production and loss terms are large, however, clouds (e.g., ‘ice clouds’ such as Cirrus clouds) can have lifetimes on the other side of days. Advection operator must not produce negative values!

Overshooting in water vapor can trigger irreversible physical processes.
Consider flux-form continuity equations for air mass and tracer mass:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (1)
\]

\[
\frac{\partial (\rho q)}{\partial t} + \nabla \cdot (\rho q \vec{v}) = 0, \quad (2)
\]

where \( \rho \) is air density and \( q \) is tracer mixing ratio. ‘Free-stream’ preservation implies that the discretization scheme for (2) reduces to (1) when \( q = 1 \).
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Relationships between long-lived stratospheric tracers, manifested in similar spatial structures on scales ranging from a few to several thousand kilometers, are displayed most strikingly if the mixing ratio of one is plotted against another, when the data collapse onto remarkably compact curves. - Plumb (2007)

E.g., when plotting nitrous oxide ($\text{N}_2\text{O}$) against ‘total odd nitrogen’ ($\text{NO}_y$) or chlorofluorocarbon (CFC’s)

It is therefore highly desirable that the transport schemes used in chemistry and chemistry-climate models should not disrupt such functional relations in unphysical ways through numerical mixing or, indeed, unmixing.’ - (Lauritzen and Thuburn, 2012)
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NCAR’s Yellowstone is a 1.5-petaflops high-performance computing system with 72,288 processor cores.

Accuracy on non-orthogonal grids (splitting errors) \(\Rightarrow\) fully two-dimensional methods

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1K 4K 16K 64K 256K
0.25 0.5 1 2 4 8
NCORES
Simulated Years/Day

Figure 5: Performance of the CESM atmosphere component model on Intrepid (IBM BG/P) when using the CAM-SE, FV or EUL dynamical core, showing the simulated-years-per-day as a function of the number of processing cores. Atmosphere component times taken from a CESM time-slice simulation, coupling the atmosphere (at 0.25° resolution), the land model (0.25° resolution), and the sea ice and data ocean model (0.1°). The solid black line shows perfect parallel scalability. When using CAM-SE, the CESM achieves near perfect scalability down to one element per processor, running at 12.2 SYPD on 86,400 cores.

Performance in through-put for different dynamical cores in NCAR’s global atmospheric climate model:
- horizontal resolution: approximately 25km × 25km grid boxes
- EUL = spectral transform (lat-lon grid)
- FV = finite-volume (reg. lat-lon grid)
- SE = spectral element (cubed-sphere grid)

Computer = Intrepid (IBM Blue Gene/P Solution) at Argonne National Laboratory

Peter Hjort Lauritzen (NCAR)
Part I
New geometrically flexible multi-tracer scheme
CSLAM is based on pioneering work by Dukowicz (1984), Ramshaw (1985), Dukowicz and Baumgardner (2000), and Margolin and Shashkov (2003)!

Finite-volume Lagrangian form of continuity equation for $\psi = \rho, \rho \phi$:

$$\int_{A_k} \psi_{k,n+1} \, dx \, dy = \int_{a_k} \psi_{k,n} \, dx \, dy = \sum_{\ell=1}^{L_k} \int \int_{a_{k\ell}} f_{\ell}(x,y) \, dx \, dy,$$

where the $a_{k\ell}$'s are non-empty overlap regions:

$$a_{k\ell} = a_k \cap A_\ell, \quad a_{k\ell} \neq \emptyset; \quad \ell = 1, \ldots, L_k. \quad (1)$$

For higher-order upstream cell-edge approximations see Ullrich et al. (2013).
Finite-volume Lagrangian form of continuity equation for $\psi = \rho, \rho \phi$:

$$\int_{A_k} \psi_k^{n+1} \, dx \, dy = \int_{a_k} \psi_k^n \, dx \, dy = \sum_{\ell=1}^{L_k} \int \int_{a_{k\ell}} f_\ell(x, y) \, dx \, dy,$$

where $\partial a_{k\ell}$ is the boundary of $a_{k\ell}$ and

$$f_\ell(x, y) = \sum_{i+j \leq 2} c_\ell^{(i,j)} x^i y^j.$$
Finite-volume Lagrangian form of continuity equation for $\psi = \rho, \rho \phi$:

$$
\int_{A_k} \psi_{k}^{n+1} \, dx \, dy = \int_{a_k} \psi_{k}^{n} \, dx \, dy = \sum_{\ell=1}^{L_k} \oint_{\partial a_{k\ell}} [P \, dx + Q \, dy],
$$

where $\partial a_{k\ell}$ is the boundary of $a_{k\ell}$ and

$$
\sum_{i+j \leq 2} \left[ -\frac{\partial P^{(i,j)}}{\partial y} + \frac{\partial Q^{(i,j)}}{\partial x} \right] = f_{\ell}(x, y) = \sum_{i+j \leq 2} c_{\ell}^{(i,j)} x^i y^j.
$$
Finite-volume Lagrangian form of continuity equation for $\psi = \rho, \rho \phi$:

$$
\int_{A_k} \psi_k^{n+1} \, dx \, dy = \int_{a_k} \psi_k^n \, dx \, dy = \sum_{\ell=1}^{L_k} \left[ \sum_{i+j \leq 2} c_{i,j}^{(i,j)} w_{k\ell}^{(i,j)} \right],
$$

where weights $w_{k\ell}^{(i,j)}$ are functions of the coordinates of the vertices of $a_{k\ell}$.

$w_{k\ell}^{(i,j)}$ can be re-used for each additional tracer $\Rightarrow$ multi-tracer efficiency!
Air density $\rho$ and tracer mixing ratio $q$ must be coupled carefully to ensure:

- mass-conservation
- shape-preservation ($q$ is invariant following parcels; not $\rho q$)
- ‘free-stream’ preservation

A ‘Lagrangian’ solution (Appendix B of Nair and Lauritzen, 2010)

- In cell $k$ reconstruct sub-grid-scale distribution for $\rho$ and $q$ separately:
  \[
  \rho(x, y) = \sum_{i+j \leq 2} \rho^{(i,j)} x^i y^j \quad \text{and} \quad q(x, y) = \sum_{i+j \leq 2} q^{(i,j)} x^i y^j.
  \]

- Apply shape-preserving reconstruction filter to $q(x, y)$ (see next slide)

- In Eulerian cell $k$ tracer mass sub-grid-scale reconstruction is:
  \[
  \rho q(x, y) = \overline{\rho_k} q_k(x, y) + \overline{q_k} \left[ \rho_k(x, y) - \overline{\rho_k} \right], \tag{1}
  \]
  where $\overline{\cdot}$ is cell average value; $\overline{q_k} = \overline{\rho q_k}/\overline{\rho_k}$

$\Rightarrow q = 1 \Rightarrow$ reconstruction (1) reduces to reconstruction of air density!

$\Rightarrow \rho q(x, y)$ is degree 2 (with $\rho q(x, y) = \rho(x, y) \times q(x, y)$ it would have been 4)
Scale the reconstruction function $\psi(x, y)$ so that extreme values lie within the adjacent cell-average values (can be applied selectively for less diffusion, Harris et al., 2010).

Note that enforcing shape-preservation is ‘harder/stricter’ than for flux-form schemes:

- For Lagrangian schemes we can’t mix low and high-order fluxes (e.g. Zalesak, 1979)
- Reconstruction functions must satisfy mass-conservation constraint:

$$\int_{A_k} \psi_k(x, y) \, dA = \bar{\psi}_k \Delta A_k,$$

where $\bar{\psi}_k$ is cell average value over $A_k$ with area $\Delta A_k$. (more on this in a moment)
Conditions for local mass-conservation (Erath et al., 2013)

- Line-integrals must span the domain without 'cracks/overlaps':

\[ \sum_{i \in \mathcal{E}} \Delta a_{ik} = \Delta A_k \quad \text{where} \quad \mathcal{E}_k = \{ \ell \mid a_{\ell k} \cap A_k \neq \emptyset \} = \{(a, b, c, d, e)\} \quad (2) \]

- 'Interior' line-integrals cancel.

- Line-integrals along boundary of Eulerian cell do not cancel since the reconstruction function is not continuous across cell boundaries.

- Boundary line-integrals must integrate \( f_\ell(x, y) \) exactly:

\[ \sum_{\ell \in \mathcal{E}_k} \left[ \sum_{i+j \leq 2} c_\ell^{(i,j)} x^i y^j \right] = \overline{\psi}_k \Delta A_k, \quad (3) \]

Satisfying (3) on sphere can be tricky - next slide!
In this Section we first discuss how the panel boundaries are treated in CSLAM. The mechanism for mass flux exchange is based on line-integrals on the sphere: gnomonic projection.

- For cells that stay completely on a panel when being transported by the flow (for one time-step) the overlap areas are computed exactly (Ullrich et al., 2009):

\[ I_{m}^{k} = \begin{cases} 
-\arctan \left( \frac{xy}{\rho} \right), & k = 0 \\
- y \arcsinh \left( \frac{x}{\sqrt{1 + y^2}} \right) - \arccos \left( \frac{x}{\sqrt{1 + x^2}} \right), & k = 2 \\
\arcsinh \left( \frac{y}{\sqrt{1 + x^2}} \right), & k = 1 \\
- x \arcsinh \left( \frac{y}{\sqrt{1 + x^2}} \right) - \arccos \left( \frac{x}{\sqrt{1 + y^2}} \right), & k = 0 \\
\arcsinh \left( \frac{x}{\sqrt{1 + y^2}} \right), & k = 1 \\
\rho, & k = 1 
\end{cases} \]

where \( R \) radius and \( \alpha, \beta = \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \) central angles.

→ some integrals ‘ill-conditioned’, in particular, at high resolution!
Line-integrals on the sphere: gnomonic projection

‘Cartesian-like’ coordinates:

\[(x, y) = R \left( \tan \alpha, \tan \beta \right) \quad (4)\]

where \(R\) radius and \(\alpha, \beta = \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right]\) central angles.

Performing line-integrals along cell boundaries with Gaussian quadrature is much more robust, however, integrals are not exact!

Enforce consistency: Locally scale weights so that

\[\sum_{i \in \mathcal{E}} a_{ik} = A_k \quad (5)\]

and similarly for higher-order moments.

(Erath et al., 2013)
Accuracy of linear transport (Lauritzen et al., 2012)

New diagnostics/test case suite designed to assess:

1. numerical order of convergence,
2. ‘minimal’ resolution,
3. ability of the transport scheme to preserve filaments,
4. ability of the transport scheme to transport ‘rough’ distributions,
5. ability of the transport scheme to preserve pre-existing functional relations between tracers,
6. ability of transport scheme to deal with divergent flows (Nair and Lauritzen, 2010).

Manuscript comparing 17 state-of-the-art schemes using new standard test case suite is almost complete (Lauritzen et al., 2013).
CSLAM implemented in CAM-SE for ‘offline’ transport (Erath et al., 2012)

- SE = spectral element dynamical core in CAM/HOMME (Dennis et al., 2012)
- SE uses elements and each element has a quadrature grid
- CSLAM uses an equi-angular gnomonic finite-volume grid

Figure: (left) ‘CSLAM grid’ and (right) spectral element quadrature grid

Infrastructure is being implemented to support coarser or finer finite-volume physics grid (physics grid may, of course, also simply coincide with CSLAM grid).
CSLAM implemented in CAM-SE for ‘offline’ transport (Erath et al., 2012)

- SE = spectral element dynamical core in CAM/HOMME (Dennis et al., 2012)
- SE max. Courant number (CN): $CN < 0.28$
- CSLAM max. Courant number for SE implementation: $CN < 1$

These performance numbers are for exact trajectories!
‘Online’ coupling CAM-SE with CSLAM - ongoing work!

Constraints: mass-conservation, consistency, shape-preservation

CAM-SE predicted $\rho_{SE}$ does, obviously, not match ‘offline’ $\rho_{CSLAM}$ computed by CSLAM!

Possible solutions:
- overwrite $\rho_{SE}$ with $\rho_{CSLAM}$; unstable?
- nudge $\rho_{SE}$ towards $\rho_{CSLAM}$; unstable?
- switch to flux-form version of CSLAM (Harris et al., 2010) and use well-known
finite-volume method for coupling: SE provides accumulated background flux of air
mass and CSLAM provides average flux of $q$ (satisfies all constraints!):

$$\text{Tracer mass flux } = \langle q \rangle_{CSLAM} \sum_{j=1}^{nsplit} \rho_{SE}^{(n+j/n\text{split})}$$
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$$\text{Tracer mass flux } = \langle q \rangle_{CSLAM} \sum_{j=1}^{n_{split}} (n+j/n_{split})$$

\[ n+2/4 \quad n+1/4 \quad n \]

\[ \text{time} \quad \text{flow direction} \]

\[ \rho \]

- $\rho_{n+2/4}$
- $\rho_{n+1/4}$
- $\rho_{n}$
- $\rho_{n+3/4}$
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\text{Tracer mass flux } = \langle q \rangle_{CSLAM} \sum_{j=1}^{n_{\text{split}}} \rho_{SE}^{(n+j/n_{\text{split}})}
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$$\text{Tracer mass flux } = \langle q \rangle_{CSLAM} \sum_{j=1}^{n_{split}} \rho_{SE}^{(n+j/n_{split})}$$
Finite-volume approach: Integrate in space

**semi-Lagrangian form**

\[
\frac{D}{Dt} \int_{A(t)} \psi \, dA = 0.
\]

where \( A(t) \) is a Lagrangian\(^\dagger \) control volume and

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla,
\]

is the material/total derivative.

---

**Eulerian (flux-form) form**

Integrate

\[
\frac{\partial \psi}{\partial t} + \nabla \cdot (\psi \, \vec{v}) = 0
\]

over an Eulerian control volume \( A_k \):

\[
\frac{\partial}{\partial t} \int_{A_k} \psi \, dA + \int_{A_k} \nabla \cdot (\psi \, \vec{v}) \, dA = 0.
\]

\(^\dagger \) volume whose bounding surface moves with the local fluid velocity ⇔ volume which always contains the same material particles
Finite-volume approach: Integrate in space

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where \(A(t)\) is a Lagrangian\(^\dagger\) control volume and \(D/Dt = \partial/\partial t + \vec{v} \cdot \nabla\), is the material/total derivative.

**Eulerian (flux-form) form**

Apply divergence theorem on second term:

\[
\frac{\partial}{\partial t} \int_{A_k} \psi dA + \oint_{\partial A_k} (\psi \vec{v}) \cdot \vec{n} dS = 0,
\]

where \(\partial A_k\) is the boundary of \(A_k\) and \(\vec{n}\) the outward normal vector to \(\partial A_k\).

→ instantaneous flux of tracer mass through boundaries of \(A_k\)

\(^\dagger\) volume whose bounding surface moves with the local fluid velocity ⇔ volume which always contains the same material particles
Finite-volume approach: Integrate in time

semi-Lagrangian form

\[ \int_{A(t+\Delta t)} \psi dA = \int_{A(t)} \psi dA, \]
where $\Delta t$ is time-step and $t = n \Delta t$.

Upstream semi-Lagrangian approach:

\[ \bar{\psi}^{n+1}_k \Delta A_k = \bar{\psi}^n_k \Delta a_k, \]
where $\bar{()}$ is average value over cell.

Eulerian (flux-form) form

Apply divergence theorem on second term:

\[ \frac{\partial}{\partial t} \int_{A_k} \psi dA + \oint_{\partial A_k} (\psi \vec{v}) \cdot \vec{n} dS = 0, \]
where $\partial A_k$ is the boundary of $A_k$ and $\vec{n}$ the outward normal vector to $\partial A_k$.

$\rightarrow$ instantaneous flux of tracer mass through boundaries of $A_k$
Finite-volume approach: Integrate in time

semi-Lagrangian form

\[
\int_{A(t+\Delta t)} \psi \, dA = \int_{A(t)} \psi \, dA,
\]
where \( \Delta t \) is time-step and \( t = n \Delta t \).

Upstream semi-Lagrangian approach:

\[
\overline{\psi}^{n+1} \Delta A_k = \overline{\psi}^n \Delta a_k,
\]
where \( \overline{()} \) is average value over cell.

Eulerian (flux-form) form

\[
\frac{\partial}{\partial t} \int_{A_k} \psi \, dA + \oint_{\partial A_k} (\psi \vec{v}) \cdot \vec{n} \, dS = 0,
\]

\[
\overline{\psi}^{n+1} \Delta A_k = \overline{\psi}^n \Delta A_k + \int_{n\Delta t}^{(n+1)\Delta t} \left[ \oint_{\partial A_k} (\psi \vec{v}) \cdot \vec{n} \, dS \right] dt = 0,
\]

\( \rightarrow \) flux of tracer mass through boundaries of \( A_k \) during \( t \in [n\Delta t, (n+1)\Delta t] \)
Finite-volume approach:

semi-Lagrangian form

\[ \int_{A(t+\Delta t)} \psi \, dA = \int_{A(t)} \psi \, dA, \]
where \( \Delta t \) is time-step and \( t = n \Delta t \).

Upstream semi-Lagrangian approach:

\[ \overline{\psi}^{n+1}_k \Delta A_k = \overline{\psi}^n_k \Delta a_k, \]
where \( \overline{} \) is average value over cell.

Eulerian (flux-form) form

\[ \overline{\psi}^{n+1}_k \Delta A_k = \overline{\psi}^n_k \Delta A_k - \sum_{\tau=1}^{4} F_k^{(\tau)}, \]
where

\[ F_k^{(\tau)} = s_k^{(\tau)} \int_{a_k^\tau} \psi^n(x, y) \, dA. \]
is flux of mass through face \( \tau \) during \( \Delta t \), and \( s_k^{(\tau)} = \pm 1 \)

for simplicity assume \( s_k^{(\tau)} \) is NOT multi-valued; for multi-valued case see, e.g., Harris et al. (2010).
Finite-volume approach:

\[
\psi_k^{n+1} \Delta A_k = \psi_k^n \Delta a_k,
\]

Note equivalence between Lagrangian cell-integrated and Eulerian flux-form continuity equations:

\[
\Delta A_k - \sum_{\tau=1}^{4} \left( s_k^{(\tau)} \Delta a_k^{(\tau)} \right) = \Delta a_k.
\]

\[
\psi_k^{n+1} \Delta A_k = \psi_k^n \Delta A_k - \sum_{\tau=1}^{4} F_k^{(\tau)},
\]

i.e. the areas involved in Eulerian forecast equals upstream Lagrangian area \( a_k \).
Finite-volume approach:

**semi-Lagrangian form**

\[
\psi_{n+1}^k \Delta A_k = \psi^n_k \Delta a_k,
\]

Define a global piecewise continuous reconstruction function

\[
\psi(x, y) = \sum_{k=1}^{N} I_{A_k} \psi_k(x, y),
\]

where \( I_{A_k} \) is the indicator function

\[
I_{A_k} = \begin{cases} 
1, & (x, y) \in A_k, \\
0, & (x, y) \notin A_k.
\end{cases}
\]

**Eulerian (flux-form) form**

\[
\psi_{n+1}^k \Delta A_k = \psi^n_k \Delta A_k - \sum_{\tau=1}^{4} F^{(\tau)}_k,
\]
Finite-volume approach:

**semi-Lagrangian form**

\[
\psi_{k}^{n+1} \Delta A_{k} = \psi_{k}^{n} \Delta a_{k},
\]

\[
\psi_{k}^{n+1} \Delta A_{k} = \sum_{\ell=1}^{L_{k}} \int_{a_{k\ell}} \psi_{\ell}^{n}(x, y) \, dA.
\]

where \( a_{k\ell} \) is the non-empty overlap area \( a_{k\ell} = a_{k} \cap A_{\ell}, \quad a_{k\ell} \neq \emptyset; \quad \ell = 1, \ldots, L_{k} \),

where \( N \) is the number of cells in the domain and \( L_{k} \) number of overlap areas.

**Eulerian (flux-form) form**

\[
\psi_{k}^{n+1} \Delta A_{k} = \psi_{k}^{n} \Delta A_{k} - \sum_{\tau=1}^{4} F_{k}^{(\tau)},
\]
Finite-volume approach:

**semi-Lagrangian form**

\[
\psi_{k}^{n+1} \Delta A_{k} = \psi_{k}^{n} \Delta a_{k},
\]

where \( a_{k \ell} \) is the non-empty overlap area \( a_{k \ell} = a_{k} \cap A_{\ell}, \quad a_{k \ell} \neq \emptyset; \quad \ell = 1, \ldots, L_{k} \),

where \( N \) is the number of cells in the domain and \( L_{k} \) number of overlap areas.

\[
\psi_{k}^{n+1} \Delta A_{k} = \sum_{\ell=1}^{L_{k}} \int_{a_{k \ell}} \psi_{\ell}^{n}(x, y) \, dA.
\]

**Eulerian (flux-form) form**

\[
\psi_{k}^{n+1} \Delta A_{k} = \psi_{k}^{n} \Delta A_{k} - \sum_{\tau=1}^{4} F_{k}^{(\tau)},
\]

where \( L_{k}^{(\tau)} \) is number of non-empty ‘flux’ overlap areas for face \( \tau \).

\[
F_{k}^{(\tau)} = \sum_{\ell=1}^{L_{k}^{(\tau)}} \int_{a_{k \ell}} \psi_{\ell}^{n}(x, y) \, dA,
\]

Note that in general: \( L_{k} \ll \sum_{\tau=1}^{4} L_{k}^{(\tau)} \)
Finite-volume approach: Conditions for inherent mass-conservation

semi-Lagrangian form

\[ \psi_{k+1}^{n+1} \Delta A_k = \psi_k^n \Delta a_k , \]

- \( a_k \)'s span \( \Omega \) without gaps/overlaps

\[ \bigcup_{k=1}^{N} a_k = \Omega , \text{ and } a_k \cap a_\ell = \emptyset \forall k \neq \ell . \]

- Sub-grid-scale representation of \( \psi \) must integrate to cell-average mass

\[ \int_{A_k} \psi_k^n(x, y) \, dA = \bar{\psi}_k^n \Delta A , \]

Eulerian (flux-form) form

\[ \psi_{k+1}^{n+1} \Delta A_k = \psi_k^n \Delta A_k - \sum_{\tau=1}^{4} F_k^{(\tau)} , \]

- Fluxes for ‘shared’ faces must cancel, e.g.,

\[ F_k^{(3)} = -F_k^{(1)} \]

Any flux, even highly inaccurate fluxes, will NOT violate mass-conservation!
Finite-volume approach: Enforcing shape-preservation

semi-Lagrangian form

Eulerian (flux-form) form

The only direct way of enforcing shape-preservation is to filter the sub-grid-scale distribution $\psi_n^k(x, y)$:

- fully 2D filters (Barth and Jespersen, 1989)
- 1D filters for cascade schemes (Colella and Woodward, 1984; Zerroukat et al., 2005; Lin and Rood, 1996)

Shape-preservation can be enforced by

- blending monotone and high-order fluxes (e.g., Flux-Corrected Transport Zalesak, 1979)
- making $\psi_n^k(x, y)$ shape-preserving (Barth and Jespersen, 1989)
Part II
Beyond linear transport: shallow-water model
Equations of motion

- Shallow water equations on an $f$-plane:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial x} - f v - g \frac{\partial h}{\partial x} = 0
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial x} + f u - g \frac{\partial h}{\partial y} = 0
\]
\[
\frac{\partial h}{\partial t} + \nabla \cdot (h \vec{v}) = 0
\]
\[
\frac{\partial (h q)}{\partial t} + \nabla \cdot (h q \vec{v}) = 0
\]

where $f$ Coriolis, $h$ height, $\vec{v} = (u, v)$ horizontal velocity vector, $g$ gravity.

- Momentum equation solved using traditional two-time-level semi-implicit scheme.
- Continuity equations solved with cell-integrated scheme: CSLAM
  - Semi-implicit time-stepping with cell-integrated Lagrangian schemes not straight forward (consistency, divergence discretization)
Traditionally: semi-Lagrangian advection of $\rho$ is combined with semi-implicit time-stepping:

$$\bar{\rho}_{n+1}^{k} = \left(\bar{\rho}_{n+1}^{k}\right)_{exp} - \frac{\Delta t}{2} \rho_{00} (\nabla \cdot \vec{v}_{n+1}^{k} - \nabla \cdot \tilde{\vec{v}}_{n+1}^{k}),$$

where

- $\rho_{00}$ a constant reference density
- $(\cdot)_{exp}$ is the explicit prediction
- $\tilde{\vec{v}}^{n+1}$ velocity extrapolated to time-level $(n + 1)$

What about tracers?

- Solving continuity equation for $(\rho q)$ explicitly

$$\bar{\rho} q_{n+1}^{k} \Delta A_k = \bar{\rho} q_{n}^{k} \Delta a_k$$

is NOT ‘free-stream’ preserving!

- Using ‘traditional’ semi-implicit approach for tracers

$$\bar{\rho} q_{n+1}^{k} \Delta A_k = \bar{\rho} q_{n}^{k} \Delta a_k - \frac{\Delta t}{2} (\rho q)_{00} \left(\nabla \cdot \vec{v}_{n+1}^{k} - \nabla \cdot \tilde{\vec{v}}_{n+1}^{k}\right).$$

is ‘free-stream’ preserving but problematic (Lauritzen et al., 2008).
Time-stepping and coupling: mass-conservative semi-implicit approach

Traditionally: semi-Lagrangian advection of $\rho$ is combined with semi-implicit time-stepping:

$$\bar{\rho}_{k}^{n+1} = (\bar{\rho}_{k}^{n+1})_{exp} - \frac{\Delta t}{2} \left\{ \nabla \cdot \left[ (\bar{\rho}_{k}^{n+1})_{exp} \bar{v}_{k}^{n+1} \right] - \nabla \cdot \left[ (\bar{\rho}_{k}^{n})_{exp} \tilde{v}_{k}^{n+1} \right] \right\}. $$

where

- $\rho_{00}$ a constant reference density
- $(\cdot)_{exp}$ is the explicit prediction
- $\tilde{v}^{n+1}$ velocity extrapolated to time-level $(n+1)$

- Radially propagating gravity wave test (shallow water in Cartesian geometry; Wong et al., 2013b)
  - Initial condition: $q = 1$
  - Errors are $O(10^{-3})$
  - Problematic? Even when using a shape-preserving filter the semi-implicit correction term may render the scheme oscillatory and non-shape-preserving!
Traditionally: semi-Lagrangian advection of $\rho$ is combined with semi-implicit time-stepping:

$$
\bar{\rho}_{k}^{n+1} = (\bar{\rho}_{k}^{n+1})_{exp} - \frac{\Delta t}{2} \left\{ \nabla \cdot \left[ (\bar{\rho}_{k}^{n+1})_{exp} \bar{v}_{k}^{n+1} \right] - \nabla \cdot \left[ (\bar{\rho}_{k}^{n+1})_{exp} \bar{v}_{k}^{n+1} \right] \right\}.
$$

where

- $\rho_{00}$ a constant reference density
- $(\cdot)_{exp}$ is the explicit prediction
- $\bar{v}_{n+1}$ velocity extrapolated to time-level $(n+1)$

What about tracers?

- A solution is to formulate the semi-implicit terms in flux-form

$$
\bar{\rho}_{q_{k}}^{n+1} = (\bar{\rho}_{q_{k}}^{n+1})_{exp} - \frac{\Delta t}{2} \left\{ \nabla \cdot \left[ (\bar{\rho}_{q_{k}}^{n+1})_{exp} \bar{v}_{k}^{n+1} \right] - \nabla \cdot \left[ (\bar{\rho}_{q_{k}}^{n+1})_{exp} \bar{v}_{k}^{n+1} \right] \right\}.
$$

so that reference states are eliminated (Wong et al., 2013b)
Traditionally: semi-Lagrangian advection of $\rho$ is combined with semi-implicit time-stepping:

$$\overline{\rho}_{k}^{n+1} = \overline{\rho}_{k}^{n+1}_{\text{exp}} - \frac{\Delta t}{2} \left\{ \nabla \cdot \left[ (\overline{\rho}_{k}^{n+1})_{\text{exp}} \overline{v}_{k}^{n+1} \right] - \nabla \cdot \left[ (\overline{\rho}_{k}^{n})_{\text{exp}} \tilde{v}_{k}^{n+1} \right] \right\}. $$

- Radially propagating gravity wave test: error measures for $q$ as a function of $\Delta t$
  - solid lines is ‘problematic’ formulation
  — dash lines is new formulation
- Initial condition: $q = 1$
- Errors in semi-implicit correction term increase with increasing $\Delta t$
- New formulation is ‘free-stream preserving’ and shape-preserving!
- Both formulations are stable for long $\Delta t$’s
In traditional semi-implicit semi-Lagrangian scheme, divergence is usually discretized with finite-differences:

\[ \nabla_{eul} \cdot \vec{v} = \frac{u_{i+1,j} - u_{ij}}{\Delta x} + \frac{v_{i,j+1} - v_{ij}}{\Delta y}, \quad (6) \]

however, cell-integrated schemes ‘see’ a Lagrangian discretization of divergence based on area change:

\[ \nabla_{lgr} \cdot \vec{v} = \frac{1}{\Delta A_k} \frac{\Delta A_k - \delta A_k}{\Delta t} \quad (7) \]
Simple example graphically illustrating difference between $\nabla_{\text{eul}} \cdot \vec{v}$ and $\nabla_{\text{lgr}} \cdot \vec{v}$

Figure: Assume the following velocity components at cell vertices

\[
\begin{align*}
\vec{v}_{SW} &= (0, 0), \\
\vec{v}_{NW} &= (0, v), \\
\vec{v}_{SE} &= (u, 0), \\
\vec{v}_{NE} &= (u, v),
\end{align*}
\]

where standard compass notation has been used.

Eulerian discretization of divergence:

\[
\nabla_{\text{eul}} \cdot \vec{v} = \frac{u}{\Delta x} + \frac{v}{\Delta x}
\]  \hspace{1cm} (6)

Lagrangian (cell-integrated) discretization of divergence:

\[
\nabla_{\text{lgr}} \cdot \vec{v} = \frac{u}{\Delta x} + \frac{v}{\Delta x} - \Delta t \frac{u v}{\Delta x \Delta y}.
\]  \hspace{1cm} (7)

→ differ by non-linear term!
Time-stepping and coupling: mass-conservative semi-implicit approach

Simple example graphically illustrating difference between $\nabla_{\text{eul}} \cdot \vec{v}$ and $\nabla_{\text{lgr}} \cdot \vec{v}$

Figure: Assume the following velocity components at cell vertices

$$
\vec{v}_{SW} = (0, 0), \quad \vec{v}_{NW} = (0, v), \quad \vec{v}_{SE} = (u, 0), \quad \vec{v}_{NE} = (u, v),
$$

where standard compass notation has been used.

To have consistency with CSLAM (use $\nabla_{\text{lgr}} \cdot \vec{v}$) and retain a Helmholtz equation for the semi-implicit solve, the continuity equation is discretized as follows

$$
\rho q_k^{n+1} = (\rho q_k^{n+1})_{\text{exp}} - \frac{\Delta t}{2} \left\{ \nabla_{\text{eul}} \cdot \left[ (\rho q_k^{n+1})_{\text{exp}} \tilde{v}_k^{n+1} \right] - \nabla_{\text{lgr}} \cdot \left[ (\rho q_k^n)_{\text{exp}} \tilde{v}_k^n \right] \right\} \\
+ \frac{\Delta t}{2} \left\{ \nabla_{\text{eul}} \cdot \left[ (\rho q_k^{n+1})_{\text{exp}} \tilde{v}_k^{n+1} \right] - \nabla_{\text{lgr}} \cdot \left[ (\rho q_k^n)_{\text{exp}} \tilde{v}_k^n \right] \right\} \frac{\delta a_k}{\Delta A_k}, \quad (6)
$$

(Lauritzen et al., 2006; Wong et al., 2013b)
Shallow-water channel on the plane: Gaussian jet (Poulin and Flierl, 2003)

- (a) Traditional grid-point method for momentum equations and CSLAM for mass
- (b) Traditional grid-point method for momentum and continuity equations
- (c) Same as (a) but with ‘problematic’ semi-implicit method (LKM)
- (d) Eulerian discretization (semi-implicit leapfrog; Asselin filter)

Figure: Vorticity
Note: even with shape-preserving filter on explicit advection the semi-implicit correction terms render solution non-shape-preserving for the ‘problematic formulation (LKM)
Part III

A compressible nonhydrostatic cell-integrated semi-Lagrangian semi-implicit solver (CSLAM-NH) with consistent and conservative transport
Two-dimensional \((x - z)\) moist Euler equations in Cartesian geometry

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{\pi}{\rho_m} \gamma R_d \frac{\partial \Theta_m'}{\partial x} + F_u, \quad (7)
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{\pi}{\rho_m} \gamma R_d \frac{\partial \Theta_m'}{\partial z} + \frac{g}{\rho_m} \left[ \bar{\rho}_d \frac{\pi'}{\bar{\pi}} - \rho_m' \right] + F_w, \quad (8)
\]

\[
\frac{\partial \Theta_m}{\partial t} + \nabla \cdot (\Theta_m \mathbf{v}) = F_\Theta, \quad (9)
\]

\[
\frac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \mathbf{v}) = 0, \quad (10)
\]

\[
\frac{\partial Q_j}{\partial t} + \nabla \cdot (Q_j \mathbf{v}) = F_{Q_j}, \quad (11)
\]

\[
p = p_0 \left( \frac{R_d \Theta_m}{p_0} \right)^\gamma, \quad (12)
\]

(Klemp et al., 2007)

- Equations linearized about hydrostatically balanced background state
- Momentum equations cast in their advective form
- All other equations (density, potential temperature, moist species, cast in their conservative flux-form).
Density current with mean background flow (Straka et al., 1992)

Wong et al. (2013a):

- Symmetric solutions!
- Stable with $2 \times$ split-explicit time-step
Kessler microphysics scheme: diagnoses ‘warm rain’ (source/sink for water vapour, cloud water, and rainwater; latent heat release adjusts potential temperature).

- Vertical velocity (colored contours); solid contour - convective cloud structure
- CSLAM-NH looks more like 5th-order ‘WRF’ solution than 2nd-order
Moisture statistics (Wong et al., 2013a)

M. Wong is currently working on ‘adding’ topography to the CSLAM-NH!


References II


