A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLAM) on the Cubed-Sphere

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Problem formulation

Consider two-dimensional transport equation (no sources/sinks):

$$\frac{d}{dt} \int_{A(t)} \psi dA = 0,$$

where $\psi$ is the density, $dA$ is the element area, and the integration is over a arbitrary Lagrangian area $A(t)$ at time $t$, that is, an area that moves with the flow with no flux through its boundaries. A temporal discretization of (1) reads

$$\int_{A(t+\Delta t)} \psi dA = \int_{A(t)} \psi dA,$$

where $\Delta t$ is time-step size. In an upstream semi-Lagrangian method $A(t+\Delta t)$ is a static grid cell (arrival cell) and $A(t)$ is a deformed (departure) cell.

Notation:

- Denote the (regular) arrival cell $A_k$ and the corresponding departure cell $a_l$, where $k = 1, \ldots, N$ and $N$ is the total number of cells in the domain $\Omega$. Their respective areas are denoted $\Delta A_k$ and $\Delta a_l$.
- Let $a_{kl}$ be the non-empty overlap region between departure cell $a_l$ and grid cell $A_k$ such that
  
  $$a_{kl} = a_l \cap A_k, \quad a_{kl} \neq \emptyset; \quad k = 1, \ldots, N, \quad 1 \leq l \leq N.$$
- The sub-grid-scale reconstruction in cell $l$ is denoted $f_l(x,y)$:
  
  $$f_l(x,y) = \sum_{i,j=1}^{p=6} c_{ij}^{(p)} \phi_{ij}(x,y), \quad i,j \in \{0,1,2\},$$

where $c_{ij}^{(p)}$ are derived coefficients ensuring mass-conservation.

Continuous scheme:

The semi-Lagrangian finite-volume version of the transport equation (1) can be written as follows:

$$\nabla_k^{-1} \Delta A_k = \nabla_k^{-1} \delta a_l,$$

where $\nabla_k^{-1}$ is the average tracer density in cell $k$ at time-level $n \times 1$, and $\nabla_k^{-1}$ is the average density in the departure cell

$$\nabla_k^{-1} = \frac{1}{\Delta a_l} \sum_{l=1}^{N} f_l(x,y) dx dy.$$

Given the overlap areas $a_{kl}$, the transport problem is effectively reduced to a remapping problem.

Discretization in Cartesian geometry

- The departure cell $a_l$ is defined by connecting the departure points (cell vertices) with straight lines.
- We convert area integrals in (5) into line-integrals by applying the Gauss-Green theorem (Dukowicz 1984):

$$\int_{a_{kl}} f_l(x,y) dx dy = \int_{a_{kl}} P dx + Q dy,$$

where $\partial A_k$ is boundary of $a_{kl}$, $P$ and $Q$ are potentials:

$$\frac{\partial P}{\partial x} = f_l(x,y), \quad \frac{\partial Q}{\partial y} = f_l(x,y),$$

- Given polynomial reconstruction function $f_l(x,y)$ in (3), the CSLAM scheme is given by

$$\nabla_l^{1/2} = \sum_{k=1}^{N} f_k(x,y) dx dy = \sum_{k=1}^{N} \sum_{i,j=1}^{p=6} c_{ij}^{(p)} \phi_{ij}(x,y),$$

where $c_{ij}^{(p)}$ are functions of the coordinate locations of the vertices of $a_{kl}$.
- Note: Integration of polynomials is exact.
- Note: Separation of weights $c_{ij}^{(p)}$ from reconstruction coefficients $c_{ij}^{(p)}$.
- Once the weights have been computed they can be reused for the integral of each additional tracer distribution.

Discretization in cubed-sphere geometry

- Consider cubed-sphere grids resulting from equi-angular gnomonic (central) projection. Note that any straight line on gnomonic projection corresponds to great-circle arc on sphere.
- Integrate polynomials with Gauss-Green’s theorem on cube as in (6):
  - Line-integrals along coordinate lines computed exactly (Ullrich et al. 2009).
  - Line-integrals along arbitrary straight lines are approximated using Gaussian quadrature.
- Bi-quadratic reconstruction functions; halo values are interpolated (4th-order) from neighboring panels to cells in outward extension of panel.

Algorithm: For each panel extend panel with halo cells (Fig.a), compute departure cells (Fig.b), limit departure cells to panel (Fig.c) and compute mass in those cells. Collect mass across edges (see Fig. below).

Results

Test 1: Solid-body advection of cosine bell across cube corners (Fig.a):

- How many Gaussian quadrature points are sufficient? 2 (see Table below).
- Is the cross-term $xy$ in the reconstruction function important? YES!

Test 2: Deformational flow test cases:

- Static vortices: Center vortices around cube corner (Fig. a). (b) Convergence plots. (c) and (d) are differences between analytic and CSLAM solution at day 6 at resolutions $N_x = 32$ and $N_y = 80$, respectively.
- Moving vortices test case recently introduced by Nair and Jablonowski (2008): (a) solution after 1/4 revolution, (b) convergence plot, (c) and (d) CSLAM and difference after one revolution at $N_x = 80$.

References


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A
Ao

CSLAM

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<th>Scheme</th>
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Figure shows convergence with degree of reconstruction polynomial.

In all tests: Better than 2nd-order convergence (Lauritzen et al. 2009).

- CSLAM is approximately one order of magnitude more accurate than Putman and Lin (2007) scheme.