A locally mass-conserving semi-Lagrangian regional NWP model with conserved quasi-Lagrangian vertical coordinates

B. Sørensen, a⇤ E. Kaas, a P. H. Lauritzen, b
a Niels Bohr Institute, University of Copenhagen
b Denmark National Center for Atmospheric Research, USA
⇤ Correspondence to: Niels Bohr Institute, University of Copenhagen
Juliane Maries Vej 30, 2100 Copenhagen, Denmark. E-mail: sorensen@gfy.ku.dk

In semi-Lagrangian models with quasi-Lagrangian vertical coordinates the Lagrangian levels are, in general, remapped to Eulerian model levels each physics time step. This remapping may introduce an undesirable tendency to smooth sharp gradients and create numerical diffusion in the vertical distribution. In a primitive equations model it is shown that this numerical mixing can be reduced by introducing Lagrangian levels persisting for multiple time steps, and by remapping only tendency information (for physical parameterizations) between the Lagrangian and Eulerian levels at each time step. Several methods have been tested to ensure that the deformation of the Lagrangian layers does not degrade the accuracy of the model. After several time steps - dynamically calculated or fixed - the Lagrangian levels are remapped to the traditional Eulerian model levels to ensure the stability of the model. The methodology is implemented using a Locally Mass Conserving Semi-Lagrangian scheme for transport in the horizontal direction. A new mass conserving 1D interpolation method for vertical interpolations, based on a combination of non-uniform cubic interpolation and the partition of unity principle, is applied. The method is accurate, mass conserving, and is shown to be highly efficient compared to the traditional approach. Copyright © 0000 Royal Meteorological Society

Key Words: Semi-Lagrangian; Mass-conserving; ALE; Enviro-HIRLAM; Floating Lagrangian vertical coordinates

Received …

Citation …

1. Introduction

Numerical weather and climate modelling is under constant development. Semi-implicit semi-Lagrangian (SISL) models have proven to be numerically efficient for both numerical weather prediction (NWP) and climate modelling, due to the ability to use long time steps. While future generations of non-hydrostatic NWP models may be formulated differently, it is likely that several climate models in a number of years from now will continue to be based on the primitive equations employing the traditional SISL technique.

Chemical/aerosol feedback mechanisms are becoming more and more relevant in NWP as well as climate models, since the biogenic and anthropogenic emissions can have a direct effect on the dynamics and radiative properties of the atmosphere (Arneth et al. 2009; Baklanov 2010; Baklanov and Korsholm 2008; Forkel et al. 2012). To include chemical feedbacks a prognostic representation of chemical species is necessary, i.e. online coupling. For the models to be able to include a large number of chemical species, the advection schemes must have a high degree of multi-tracer efficiency, e.g. Lauritzen et al. (2010).

In this paper a new semi-Lagrangian (SL) advection method is introduced. It combines an inherently mass conserving 2D semi-Lagrangian advection scheme with semi-implicit time-stepping and employing a fully Lagrangian
vertical coordinate (LVC). By Lagrangian vertical coordinate is understood that the pressure levels computed during a time step are material surfaces, thus conserving mass in the corresponding Lagrangian layers. This minimizes the vertical diffusion and thus potentially improves the simulation of the corresponding vertical profiles of moisture, clouds, and chemical constituents (Heikes et al. 2006). Since the Lagrangian levels suffer from traditional Lagrangian limitations caused by the convergence and divergence of the flow, remappings to the Eulerian model levels are generally still required, unless dynamic remapping methods are used. However, it turns out that this need only be applied after a number of time steps. Several different remapping methods are considered and a new mass conserving interpolation routine has been created. This is a 1D non-uniform cubic interpolation combined with a partition of unity principle, which modifies the interpolation weights. The new remapping is shown to be both mass conserving, consistent, and multi-tracer efficient.

The concept of a quasi-Lagrangian approach is not new. It was introduced by Starr (1945) on a theoretical level and since then several implementations of LVC in numerical models has been carried out, e.g. Lin (2004); Lauritzen et al. (2008); Nair et al. (2009). These implementations have, however, generally not preserved the levels for more than one physics time step. The implementation presented here takes a different approach by preserving the Lagrangian levels for multiple physics time steps, either in a dynamic (adaptive) manner or fully preserved for a limited number of time steps.

The paper is organized as follows: In section 2 and 3 the traditional semi-implicit semi-Lagrangian and the locally mass conserving semi-Lagrangian methods is briefly introduced, respectively. In section 4 the new Lagrangian vertical coordinate is explained in detail, and in section 5 the different strategies for controlling the Lagrangian flow are described as well as the new interpolation routine. In section 6, 7, 8, and 9 the different setups are tested and analysed. Finally some concluding remarks in section 11.

2. Traditional semi-implicit semi-Lagrangian formulation

In a primitive equations model both Rossby- and gravity waves are supported. A traditional technique for eliminating the numerical instability associated with mainly external gravity modes is semi-implicit time differencing. The explicit semi-Lagrangian continuity equation can, for a two-time level discretization, be written as,

\[ \psi^{n+1}_{SL-exp} = \{\psi + 0.5\Delta t\psi \nabla \cdot \mathbf{v}\}^n + 0.5\Delta t\psi \nabla \cdot \mathbf{v} \]  

(1)

where \(\{\cdot\}\) denotes a value that has been interpolated to an upstream departure point and \(\{\cdot\}\) here and below denotes a value that has been extrapolated from time levels \(n - 1\) and \(n\):

\[ (\cdot)^{n+1} = 2(\cdot)^n - (\cdot)^{n-1} \]  

(2)

Following Temperton and Staniforth (1987) the corresponding semi-implicit semi-Lagrangian (SISL) continuity equation is,

\[ \psi^{n+1}_{SL} = \psi^{n+1}_{SL-exp} + \frac{\Delta t}{2}(L^1_{\psi} + \tilde{L}^1_{\psi}) \]  

(3)

where \(L_{\psi}\) and \(\tilde{L}_{\psi}\) are the linear and non-linear parts of the right-hand side of the prognostic equation for \(\psi\). In SISL the linear part is treated implicitly and the non-linear part explicitly via the extrapolation.

3. Locally Mass Conserving Semi-Lagrangian scheme

The Locally Mass Conserving Semi-Lagrangian scheme (LMCSL) by Kaas (2008), is a semi-Lagrangian method for solving the volume density continuity equation. It is based on a classical 2D semi-Lagrangian advection scheme with e.g. cubic Lagrange polynomial interpolation. However, to ensure mass conservation the divergence term in the continuity equation is replaced by modified interpolation weights. The term local mass conservation refers to the inherent ability of a scheme to conserve mass along characteristics.

One can rewrite the traditional explicit semi-Lagrangian scheme (1) as

\[ \psi^{n+1}_{k,SL-exp} = \sum_{l=1}^{K} w_{k,l} \left[ \psi^n_l - \frac{\Delta t(\psi \nabla \cdot \mathbf{v})^n_l}{2} \right] 

- \frac{\Delta t(\psi \nabla \cdot \mathbf{v})^{n+1}_k}{2} \]  

(4)

with \(k\) being an Eulerian (arrival) grid point index, \(K\) the total number of Eulerian grid points, and \(w_{k,l}\) the interpolation weight from an upstream Eulerian grid point \(l\) (at time level \(n\)) to the Lagrangian departure point \((\cdot)^{n+k}_l\). Thus, the \(w_{k,l}\) values define the SL remapping required for each Eulerian arrival point. It should be noted that these weights are independent of the actual field values and, once calculated, can be reused for all prognostic fields. For any traditional SL scheme we have,

\[ \sum_{l=1}^{K} w_{k,l} = 1. \]  

(5)

This does however not guarantee mass conservation.

3.0.1. Modified interpolation weights

As shown in Kaas (2008) formal mass conservation can be achieved by modifying the interpolation weights and skipping the divergence terms in (4). The modified weights \(\tilde{w}_{k,l}\) are defined as

\[ \tilde{w}_{k,l} = \frac{A_l}{A_k} \frac{w_{k,l}}{\sum_{m=1}^{K} w_{m,l}} \]  

(6)

which gives

\[ \tilde{w}_{k,l} = \psi^n_{k+\cdot} \equiv \sum_{l=1}^{K} \tilde{w}_{k,l} \psi^n_l \]  

(7)

with \(w_{k,l}\) being the traditional SL-weights. The LMCSL scheme forecasts the grid cell averages and not the grid point values, as the traditional SL scheme, hence the bars indicating cell averages. \(A_l\) and \(A_k\) are, respectively, the areas or, in case of full 3D, the volume covered by the \(l\)th and \(k\)-th Eulerian grid cells. It can easily be shown, that with this approach the modified weights ensure that the total
mass given off by any Eulerian grid cell to all surrounding departure points, is equal to the actual mass represented by that particular grid cell. Thus, mass conservation in the LMCSL scheme is obtained via a partition of unity principle. We will in the following continue to use the term departure point although, in reality, it is the location, at time level \( n \), of the Lagrangian point arriving, at time level \( n + 1 \), at the centroid of an Eulerian cell.

The computation of the modified weights is straightforward: it requires a single additional loop over all grid cells, where all the SL-weights are summed in an array, thus getting the denominator in (6). When all modified weights have been calculated they can be reused for any prognostic density variable. This makes the method very efficient if a large number of tracer variables is introduced, since each additional variable does not require any new computations of modified weights.

The underlying \( w_{k,l} \)-weights can be determined by different types of interpolation, and in the present implementation several are used, i.e. bilinear, biquadratic, bicubic, and mixed bilinear/bicubic interpolation. The locality of the scheme is dependent on the order of interpolation - as any traditional SL scheme the higher accuracy the less locality. The locality of the LMCSL method is the same as the traditional SL-method. With bicubic interpolation 16 cell values are used to determine the value at any departure point. Figure 1 illustrates the grid cells used to compute the value in four arrival cells. The figure displays convergence as the departure points are spaced further apart than the surrounding Eulerian grid cell centroids.

![Figure 1. Illustration of the upstream bi-cubic interpolation. 16 cell values are used to determine the value at each Lagrangian departure point. The four cells to the right are the Eulerian arrival cells, and the corresponding areas to the left are the cells used in the bicubic interpolation.](image1)

The effects of divergence are implicitly included in the modified weights and following Kaas (2008) divergence in LMCSL is logically defined as:

\[
D^{n+1/2}_{LM} = \frac{1}{\Delta t} \left( 1 - \sum_{l=1}^{K} w_{k,l} \right). \tag{8}
\]

If the sum of the modified weights is almost equal to one, meaning that the departure points are distributed equally, the divergence will be close to zero. The actual discretizations of the LMCSL scheme will be described further in section 4.

### 3.1. Shape preservation

To further improve the LMCSL scheme two locally mass conserving and shape-preserving filters have been implemented. The LMCSL scheme combined with these filters will be referred to as LMCSL-M and LMCSL-MS, with \( M \) indicating monotonic and \( S \) simple. The LMCSL-M filter makes an a posteriori anti-diffusive correction, where the mass is redistributed towards target values under monotonic and mass conserving constraints. With the anti-diffusive method the filter actually improves the accuracy of the scheme. See Kaas and Nielsen (2008) for further details. The LMCSL-MS filter is not anti-diffusive, but only redistributes the mass under mass conserving and shape preserving or positive definite constraints. To achieve the mass conserving properties of the LMCSL scheme in an operational model, a positive definite filter such as the \( M \) or \( S \) is essential, since the negative undershoots generated by all higher order methods are non-physical and must be removed using a mass conserving method (Kaas and Nielsen 2008).

The simple filter uses an iterative approach to redistribute the mass. When the filter encounters a value outside the minimum or maximum values, it will, within their maximum and minimum limits, remove or add mass to the neighbor cells\(^\ast\). It will do this relative to the distributable mass present in the cells, to ensure that no new over or under-shoots is created. If the mass is not fully distributed in the nearest cells, the next set of neighbors will be used, and so on, until all mass is distributed.\(^\dagger\) The filter is strictly a posteriori, and only depends on given maximum and minimum values, and is in this sense not limited to any particular advection method.

### 4. Lagrangian vertical scheme

The Lagrangian vertical scheme (LMCSL-LL) is an attempt to transform the traditional 3D interpolating semi-Lagrangian advection scheme into a more Lagrangian scheme regarding vertical transport. The LMCSL-LL scheme is implemented in the primitive equations model HIRLAM (Undén et al. 2002), but we note that there

\(^\ast\)By neighbor cells is meant the cells surrounding the cell in question, that is 2 cells for one dimensional distribution, 8 for two dimensional, and 27 for three dimensional.

\(^\dagger\)The radius of distribution will in general be of the same size as the stencil of the interpolation method.
are no principle obstacles implementing it into fully non-
hydrostatic models.

The present implementation takes its starting point
in a special mass-conserving version, here termed CISL-
HIRLAM, of HIRLAM. In CISL-HIRLAM, described
in detail in Lauritzen et al. (2008), horizontal transport
including the effects horizontal divergence is based on the
inherently mass conserving cell integrated semi-Lagrangian
(CISL) scheme, (Nair and Machenhauer 2002; Lauritzen
et al. 2006), while vertical transport is dealt with using a
quasi-Lagrangian vertical coordinate in a fashion similar to
that in Lin (2004). The main difference between the new
LMCSL-LL scheme and CISL-HIRLAM is that in LMCSL-
LL the horizontal transport is based on LMCSL instead
of CISL and, more fundamentally, in LMCSL-LL full
(one-dimensional) vertical remappings from the Lagrangian
to the Eulerian vertical coordinate are only performed at
intervals covering several time steps instead of at every time
step in CISL-HIRLAM.

Briefly, the general LMCSL-LL procedure is as
follows: after having calculated horizontal trajectories using
an iterative procedure equivalent to that in Lauritzen et al.
(2008), horizontal transport is performed based on the
LMCSL scheme. At the arrival column new pressures of
each Lagrangian vertical level is defined via the hydrostatic
equation. At the end of the time step the concentrations at
Lagrangian levels are stored so they can be used as a starting
point for the following time step. Because the physical
parameterization of the model is designed to operate on
fixed Eulerian levels, a one-dimensional vertical remapping
to the Eulerian levels must still be carried out at each time
step. Once calculated there, the tendencies of the physics,
including the effects horizontal divergence is based on the
hybrid sigma-pressure vertical coordinate. Between the half
levels the effect of horizontal accelerations is included
velocities in the CISL-HIRLAM case. To acquire accurate
trajectories the effect of horizontal accelerations is included
as they were in CISL-HIRLAM. This was shown, in a
shallow water model, to be of importance for the numerical
stability when the LMCSL scheme is combined with the
semi-implicit scheme (Kaas 2008) in essentially the same
way as we will present below for the LMCSL-LL scheme.

The flow between the time steps is identical to that in
Figure 3 but the Lagrangian levels are re-used in the
following time steps, and therefore they are generally
located at different pressure levels than those in Figure 3.
The solid (green) arrows represent the remapping of
equations from Eulerian levels to Lagrangian levels.

The Eulerian grid of the LMCSL-LL scheme is
identical to that in HIRLAM, i.e. the horizontal grid is
a traditional staggered Arakawa C-grid (Mesinger and
Arakawa 1976), while in the vertical a staggered Lorenz-
grid is used - see Undén et al. (2002) for details. The model
atmosphere is divided into K levels, with the lowest index
at the top of the atmosphere and the highest at the surface.
The pressure \( p_{k+1/2} \) at the interface between levels \( k \)
and \( k + 1 \) is:

\[
p_{k+1/2} = A_{k+1/2} + B_{k+1/2} p_s
\]  

where \( p_s \) is the surface pressure, and \( A_{k+1/2} \) and \( B_{k+1/2} \)
are predefined constants defining the terrain following
hybrid sigma-pressure vertical coordinate. Between the half
levels \( k - \frac{1}{2} \) and \( k + \frac{1}{2} \) are the full levels \( k \). The pressure
level thickness of level \( k \) is defined by

\[
\Delta p_k = p_{k+1/2} - p_{k-1/2}
\]  

The uppermost half level is \( p_{1/2} = 0 \) and the surface
is at \( p_{K+1/2} = p_s \).

4.1. Trajectory calculations

As mentioned above the first step in the new LMCSL-
LL scheme is to determine the trajectories. This is done
iteratively using two iterations, and in a way quite similar
to that in CISL-HIRLAM, see Lauritzen et al. (2008) for
details, although the LMCSL-LL calculation of upstream
horizontal trajectories is based on velocities originally
defined in the new Lagrangian vertical levels, while they
were based on the traditional level Eulerian horizontal
velocities in the CISL-HIRLAM case. To acquire accurate
trajectories the effect of horizontal accelerations is included
as they were in CISL-HIRLAM. This was shown, in a
shallow water model, to be of importance for the numerical
stability when the LMCSL scheme is combined with the
semi-implicit scheme (Kaas 2008) in essentially the same
way as we will present below for the LMCSL-LL scheme.

Figure 3. Flow of the traditional LMCSL scheme with remapping to
Eulerian model levels at every time step. The four columns are respectively
time step \( n \), \( n + 1 \), \( n + 2 \) and \( n + 3 \). Solid lines indicate Eulerian levels
and dashed Lagrangian levels.

Figure 4. Flow of the LMCSL-LL scheme with remapping to Eulerian
model levels at every time step, and tendency remapping to Lagrangian
levels. The four columns are respectively time step \( n \), \( n + 1 \), \( n + 2 \) and
\( n + 3 \).

Figure 4. Flow of the LMCSL-LL scheme with remapping to Eulerian
model levels at every time step, and tendency remapping to Lagrangian
levels. The four columns are respectively time step \( n \), \( n + 1 \), \( n + 2 \) and
\( n + 3 \).
Note that in the LMCSL-LL scheme trajectories relevant for solving the continuity equation are those ending up in the Eulerian mass points of the Arakawa C-grid, whereas for the CISL-L scheme trajectories were calculated for each corner point of the Eulerian control volumes. This means that for the LMCSL-LL scheme velocities must be interpolated to the mass points, defined at the center of the Eulerian cells, prior to the calculation of trajectories, instead of an interpolation to the corner points.

As for the CISL-HIRLAM two iterations have been used to estimate the trajectories. For each guess of horizontal trajectories, i.e. for each iteration, an explicit upstream LMCSL interpolation is used to obtain the updated contributions to the mass field in the arrival column. In this column the mass field is then summed from the top of the atmosphere \((p = 0)\) to the surface \((p = p_s)\) to obtain the pressure of each Lagrangian level relevant for the actual iteration. These levels do in general not coincide with the model levels defined at time step \(n\). To calculate the horizontal velocities extrapolated to time level \(n + 1\) which is needed for the following iteration and for the final explicit forecast, the trajectory is therefore interpolated vertically using cubic Lagrange interpolation to the actual Lagrangian arrival level.

### 4.2. Discretized equations of motion

The semi-implicit LMCSL-LL continuity equation is given as,

\[
\partial p_{k,L}^{n+1} + \frac{\partial}{\partial p} \left( \frac{\partial p}{\partial n} \right) = \Delta t \frac{\Delta p_{ref,k,L}^{ref}}{2} \left[ D_{k,L}^{n+1} - D \left( \nabla \hat{v}^{n+1} \right) \right] \tag{11}
\]

where \(\Delta p_{ref,k,L}^{ref}\) is the Lagrangian reference pressure level thickness defined in section 4.4. The last term on the right-hand side is the extrapolated LMCSL divergence, defined as (8),

\[
D_{k,L}^{n+1} = \frac{2}{\Delta t} \left( 1 - \sum_{l=1}^{K} w_{k,l} \right) \tag{12}
\]

but based only on extrapolated velocities. It is obtained by calculating the modified weights from the extrapolated half of the trajectory and multiplying with two, as it is only a half-trajectory. \(D\) is the Eulerian divergence discretized with finite differences.

\[
D = \frac{1}{a \cos \theta} \left[ \frac{\partial}{\partial \lambda} (a \cos \theta) \hat{u} + \frac{\partial}{\partial \theta} \hat{v} \right] \tag{13}
\]

The momentum equations are solved on semi-Lagrangian grid point form, using a half-implicit-Coriolis scheme as is used in HIRLAM. Again the formulation is the same as for the CISL-HIRLAM, see Lauritzen et al. (2008) for the details. Where LMCSL, and therefore LMCSL-LL, differs from the CISL method is the solution to the continuity equation (11). In CISL-HIRLAM the continuity equation is solved using the predictor-corrector method (Lauritzen et al. 2006). In LMCSL the Lagrangian divergence (12) is used directly to solve the continuity equation as in Kaas (2008), resulting in a somewhat simpler scheme.

### 4.3. Consistent Omega-P

In many NWP models the discretization of the thermodynamic equation is inconsistent, as is also the case for Enviro-HIRLAM and HIRLAM. The energy conversion term is there solved with an Eulerian approach using finite differences, and vertical velocities are diagnostic and calculated from the continuity equation, which is discretized in a semi-Lagrangian manner.

Using the approach by Lauritzen (2005) in the implementation of the LMCSL-LL scheme, where omega-P is defined as

\[
\left( \frac{w}{p} \right)_{exp} = \frac{2}{\Delta t} \frac{\left( \frac{\Delta p_{n+1}^{ref}}{\Delta t} \right)}{\left( \frac{\partial p}{\partial n} \right)_{n+1}} \tag{14}
\]

the vertical velocity and the energy conversion term becomes fully consistent.

### 4.4. Modified pressure level thicknesses

The traditional reference pressure level thicknesses \(\Delta p_{ref}^{ref}\) does not apply to the Lagrangian levels, and must be modified. This is done in a completely consistent way with the traditional reference pressure level thicknesses. The reference pressure level thickness is defined as the Eulerian pressure level thickness with a surface pressure of 1013 hPa. This is very simple in the hybrid coordinate model levels, since the pressure levels - and therefore the layer thicknesses - are calculated directly from the surface pressure using (9) and (10).

The change from pressure levels to reference pressure levels, is a change in surface pressure. The Lagrangian reference pressure level thickness, is then

\[
\Delta p_{ref,k,L} = \Delta p_{k,L} + \Delta B_k \Delta p_s \tag{15}
\]

where

\[
\Delta p_s = p_{ref}^{ref} - p_s \tag{16}
\]

\[
\Delta B_k = B_{k+1/2} - B_{k-1/2} \tag{17}
\]

This can easily be verified by substituting \(\Delta p_{ref,k,L}^{ref}\) with \(\Delta p_k\) and using (9) then,

\[
\Delta p_{ref,k,L} = \Delta p_k + \Delta B_k \Delta p_s
\]

\[
= \left[ (A_{k+1/2} + B_{k+1/2} p_s) - (A_{k-1/2} + B_{k-1/2} p_s) \right] + (B_{k+1/2} - B_{k-1/2}) (p_{ref}^{ref} - p_s)
\]

\[
= (A_{k+1/2} - A_{k-1/2}) + (B_{k+1/2} - B_{k-1/2}) p_{ref}^{ref}
\]

\[
= (A_{k+1/2} + B_{k+1/2} p_{ref}^{ref}) - (A_{k-1/2} + B_{k-1/2} p_{ref}^{ref})
\]

\[
= \Delta p_{ref}^{ref} \tag{18}
\]

completes the demonstration.

The surface pressure changes slightly during the physics, which is in Eulerian model space, and the Eulerian pressure level thicknesses change accordingly. The sum of the pressure level thicknesses must be equal to the surface pressure.

\[
p_s = \sum_{k=1}^{K} \Delta p_k \tag{19}
\]
The Lagrangian pressure level thicknesses are preserved, and should ideally not change. This is however not possible, since their sum will be different than the updated surface pressure. To modify the Lagrangian pressure level thicknesses in such a way that they are consistent with the updated surface pressure can be done in many ways, but it must be coherent with the underlying hybrid levels. By coherent it is understood, that in the special case where the Lagrangian levels are identical to the Eulerian levels, the updated Lagrangian levels must also be identical to the updated Eulerian levels. This can easily be done by substituting $p_s$ with $p_s^0$ and $p_{s+1}^e$ with $p_{s+1}^1$ in (15) thus getting,

$$
\Delta p_{k, L} = \Delta p_{k, E} + \Delta B_k \Delta p_s
$$

(20)

with

$$
\Delta p_s = p_s^1 - p_s^0
$$

(21)

where $p_s^0$ is the pressure before the physics and $p_s^1$ is the modified pressure after the physics.

### 4.5. Tendencies

When the full semi-implicit forecast has been performed, the forecasted fields are at Lagrangian pressure layers. These fields are saved in temporary Lagrangian variables, in addition to the traditional prognostic variables the pressure level thicknesses are kept as well, as they define the Lagrangian pressure levels.

$$
p_{k+1/2, L} = \sum_{i=1}^{k} \Delta p_i
$$

(22)

A full remapping of all variables from the Lagrangian pressure levels, to the Eulerian pressure levels is performed, as is done traditionally in the LMCSL scheme. The velocities and temperature are remapped using 1D cubic interpolation and the scalar fields are remapped with the 1D cubic LMCSL interpolation with a shape-preserving filter\(^1\). The fields are then stored before the model physics is performed. After the model physics and boundary relaxation, the new Eulerian fields at time level $n+1$ are known. The tendencies can then be calculated.

$$
\phi_{k}^{t, E} = \phi_{k}^{0} - \phi_{k}^{1}
$$

(23)

where $\phi_{k}^{t, E}$ is the tendency at Eulerian levels, $\phi_{k}^{0}$ are the remapped Eulerian values before the physics and $\phi_{k}^{1}$ is the Eulerian values after the physics.

The tendencies are remapped from the Eulerian levels back to the Lagrangian pressure levels. There is no monotonicity constraints applied to the tendencies, which are negative as well as positive. However, as the tendencies are added to the Lagrangian fields, negative values can be observed, and it is thus necessary to run the simple filter, here in one dimension and with only a positive definite constraint.

---

\(^1\)Alternatively the piecewise parabolic method (PPM2) with a monotonic constraint can be used.

### 4.6. LAM boundaries

The model system is a limited area model (LAM), it is therefore necessary to acquire boundary data from an external source such as nesting in a larger domain or from a global model. In the boundary zone surrounding the model domain, the levels are always Eulerian. This means that the built in boundary relaxation routine can be used, and no tendencies are calculated. A simple linear relaxation of the Lagrangian levels towards the Eulerian model levels is used close to the boundaries.

### 5. Remapping methods

Conserving the Lagrangian levels indefinitely is not possible, since fully Lagrangian methods face several problems. Just as in the horizontal, the vertical flow will cause the levels to converge and diverge with time, until they are spaced either too close to each other or too far from each other. The solutions presented here is either to use a limited number of Lagrangian time steps or to use a dynamic remapping method.

#### 5.1. Fixed remapping

A variable called the Lagrangian time step (LTS) has been introduced, it is defined as the number of fully Lagrangian time steps taken, before the model will continue, not from the Lagrangian levels, but from the Eulerian model levels. Since the Lagrangian fields already have been remapped to Eulerian levels, all computations has already been carried out, so what is done in praxis, is that all Lagrangian fields and variables are overwritten with the Eulerian fields. This is a very simple, but effective and highly variable method, it will even as LTS approaches zero converge toward the underlying semi-Lagrangian model, and if LTS is equal to zero, the scheme is identical to the standard LMCSL scheme as it should. This remapping method is very fast since all fields already exists, and the tendency interpolation is not needed.

#### 5.2. Dynamic remapping

In addition to the fixed remapping several different kinds of dynamic remapping schemes have been tried. The
An NWP model with quasi-Lagrangian vertical coordinates

Dynamic remapping only remaps individual levels if they violate certain conditions. This can reduce the number of remappings in some areas, but remapping in other areas might be increased compared to the fixed remapping.

**Method I**

The simplest dynamic remapping, that ensures the Lagrangian levels do not deviate excessively from the Eulerian levels, only has one condition which must not be violated. A particular Lagrangian level, may not deviate more than a given distance (in pressure coordinates), from the surrounding Eulerian levels. E.g. \( \delta = \frac{1}{3} \) where \( \delta \) is normalized pressure level distance.

\[
p_{k+1/2} - \delta \Delta p_k < p_{k+1/2} < p_{k+1/2} + \delta \Delta p_k
\]

(24)

If the Lagrangian level violates this, the level is remapped to the corresponding Eulerian level. Since the pressure levels in question is the half levels \( k + \frac{1}{2} \), and the prognostic values are defined at full levels, the full level value \( k \) and \( k + 1 \) must be remapped. This method

![Figure 6](image.png)

Figure 6. Flow of the LMCSL-LL scheme with dynamic remapping method I at individual levels, and tendency remapping to Lagrangian levels. The four columns are respectively time step \( n, n + 1, n + 2 \) and \( n + 3 \).

has some disadvantages, mainly that when a Lagrangian level is remapped, the horizontally surrounding Lagrangian levels are in general not remapped. This can introduce discontinuities in the flow. To prevent this from affecting the stability of the model, only minor deviations can be tolerated, such as \( \delta_1 \leq \frac{1}{3} \). In practice, it also adds some constraints on the time step, which cannot be too long, as it will generate larger deviations and the discontinuities increases.

**Method II**

To address the issues in method I, a simple solution is to remap levels which violate the condition, not to the corresponding Eulerian level, but instead to a new Lagrangian level which is equal to the allowable deviation. This guarantees that there is no horizontal discontinuities. The disadvantage of this method is, that the flow will in fact not be the "real" Lagrangian flow but only an approximation. Also since the new level is located at the actual boundary condition, it is highly likely that within a few time steps it will cross the boundary once more and require an additional remapping. Thus increasing the total number of remappings considerably. The remappings from one Lagrangian level to another Lagrangian level is in essence a full remapping, and will not be any more desirable than full remappings to Eulerian levels. This remapping will however be confined to a single level and not the full field.

![Figure 7](image.png)

Figure 7. Flow of the LMCSL-LL scheme with dynamic remapping method II at individual levels, and tendency remapping to Lagrangian levels. The four columns are respectively time step \( n, n + 1, n + 2 \) and \( n + 3 \).

This approach allows the Lagrangian levels to deviate much more than method I and II. This does however also allow multiple layers to occupy the space where only one Eulerian layer is present (and vice versa), i.e. greatly enhancing accuracy in that area and decreasing the accuracy where only one Lagrangian layer spans multiple Eulerian layers. Therefore the minimum layer thickness should be chosen with care.

**Method III**

The optimal method for remapping based on the convergence and divergence of the Lagrangian flow, is however to demand that the thickness of a Lagrangian layer must not become too small or too large.

\[
\delta_1 \Delta p_k < \Delta p_{k,L} < \delta_2 \Delta p_k
\]

(25)

where \( \delta_1 = \frac{2}{3} \) could be used. In practice it has not been neccessary to set an upper limit, i.e. \( \delta_2 \) is set to a large number and only the lower constraint is enforced. Using

![Figure 8](image.png)

Figure 8. Flow of the LMCSL-LL scheme with dynamic remapping method III at individual levels, and tendency remapping to Lagrangian levels. The four columns are respectively time step \( n, n + 1, n + 2 \) and \( n + 3 \).
Combined remapping

The fixed remapping method and the dynamic methods can also be combined. When using dynamic remapping, it is not necessary to remap the entire field to Eulerian levels using the LTS, but it can be done as well. A balance could possibly be obtained between an almost fully Lagrangian flow and the traditional quasi-Lagrangian flow.

5.3. Consistent vertical remapping

To further increase the consistency and the efficiency of the scheme, a new vertical remapping routine has been developed. The routine uses 1D cubic Lagrange interpolation. An equidistant Lagrangian polynomial, such as the one used by Durran (2010), creates a third-order polynomial, matching at the closest four grid points. In the vertical the grid is defined in pressure coordinates, which are irregularly spaced and changing through time. A corresponding non-uniform cubic Lagrange polynomial can be described by the following expression:

\[
\phi_* = -\frac{\lambda_1 \lambda_2 \lambda_6}{\delta_1 \delta_2} \phi_{i-2} + \frac{\alpha \lambda_1 \lambda_2}{\delta_1 \lambda_4 \lambda_7} \phi_{i-1} + \frac{\alpha \lambda_1 \lambda_6}{\delta_2 \lambda_3 \lambda_7} - \frac{\alpha \lambda_2 \lambda_6}{\lambda_3 \lambda_4 \lambda_{i+1}} \tag{26}
\]

with

\[
\begin{align*}
\lambda_1 &= 1 - \alpha \\
\lambda_3 &= 1 - \delta_2 \\
\lambda_4 &= 1 - \delta_1 \\
\lambda_2 &= \delta_2 - \alpha \\
\lambda_6 &= \alpha - \delta_1 \\
\lambda_7 &= \delta_2 - \delta_1 
\end{align*}
\]

where \(\alpha\) is the pressure distance between the grid point \(\phi_{i-2}\) and \(\phi_*\), which is the "departure" point. \(\delta_1\) is the distance between the grid point \(\phi_{i-2}\) and \(\phi_{i-1}\), and \(\delta_2\) is the distance between the grid point \(\phi_{i-2}\) and \(\phi_i\). These distances are all normalized by dividing with the distance between the outer grid points \(\phi_{i-2}\) and \(\phi_{i+1}\). Equation (26) reduces to the traditional formulation if the grid is equidistant, and is thus fully consistent with the bi-cubic horizontal interpolations, and the previously used cubic vertical interpolation for \(u, v, \) and \(T\). The original vertical interpolation routine is formulated as a Newton polynomial thus giving the same result but without the possibility for re-using interpolation weights for multi-tracer efficiency. When formulated as a Lagrange polynomial it also enables us to acquire mass conservation using the general LMCSL approach described in section 3.

\[
\hat{w}_k = \frac{w_k}{\sum_{l=1}^N w_l} \tag{27}
\]

This method will generate over- and under-shoots as any traditional higher-order interpolation, and should be used in combination with a filter if the solution must be shape preserving. This method does, however, mean that the actual remapping of tendencies is fast and efficient, in particular for multiple tracers. A comparison between the efficiency of the interpolations methods can be seen in 9.

In Figure 9 is shown four curves, which all matches the four grid values, \(\phi_{i-2} = 8, \phi_{i-1} = 3, \phi_i = 5, \phi_{i+1} = 2\).

Figure 9. Example of cubic non-uniform representation. The solid (black) curve has \(\delta_1 = \frac{1}{5}\), the dot-dashed (red) has \(\delta_1 = \frac{1}{4}\), the dotted (blue) curve has \(\delta_1 = \frac{1}{3}\), and the dashed (green) curve has \(\delta_1 = \frac{1}{2}\).

The solid (black) curve is the equidistant polynomial, as created by the traditional cubic interpolation. The remaining curves show the polynomial when the grid is non-uniform. The dot-dashed (red) curve has \(\delta_1 = \frac{1}{5}\), the dotted (blue) curve has \(\delta_1 = \frac{1}{4}\), and the dashed (green) curve has \(\delta_1 = \frac{1}{3}\). The interpolation method will be referenced as LMCSL-NU (non-uniform).

6. 1D test

A simple one dimensional advection test has been performed to compare the new non-uniform LMCSL interpolation with the traditional methods, i.e. SL and PPM2. The test is a solid body rotation on two non-uniform grids. The two grids have a relatively large ratio between the smallest and the largest cell\(^5\). In grid A the cells are smoothly varying in size and in grid B the cells vary sharply between two sizes. The smoothly varying grid A is defined by

\[
\Delta x_i = 1 + r \sin \left( \frac{2\pi(i-1)}{N} \right) \tag{28}
\]

and the sharply varying grid B is defined by,

\[
\Delta x_i = \begin{cases} 
0.5 & i \text{ even} \\
1.5 & i \text{ odd} 
\end{cases} \tag{29}
\]

where \(\Delta x_i\) is the grid cell size of the \(i\)-th grid cell and \(N\) is the number of grid cells. \(r = \frac{\text{ratio-1}}{\text{ratio}}\) is the ratio factor. The ratio used in the test is 1.5. In test case A the velocity is set to one in the entire domain, giving 100 steps per revolution. In test case B the velocity is 1.25 since the departure points would otherwise coincide with the actual grid points\(^6\) (giving exact results).

Error analysis

To quantify the errors, the following common methods of estimating errors is used.

\(^5\)The cell size ratio in the Eulerian vertical grid is approximately 1:1.4 with only 1:1.1 between neighbor cells. The Lagrangian grid can aquire much larger grid ratios, which is the reason that this test uses larger ratios.

\(^6\)We are forecasting cell averages and the arrival point is in the center of each cell.
An NWP model with quasi-Lagrangian vertical coordinates

\[ l_1 = \frac{\sum_{k=1}^{K} |\phi_k - \psi_k|}{\sum_{k=1}^{K} |\psi_k|} \]  \\
\[ l_2 = \sqrt{\frac{\sum_{k=1}^{K} (\phi_k - \psi_k)^2}{\sum_{k=1}^{K} |\psi_k|^2}} \]  \\
\[ l_\infty = \frac{\max |\phi_k - \psi_k|}{\max |\psi_k|} \]

where \( \phi_k \) is the simulation value and \( \psi_k \) is the analytic or true value.

Table I and II show that both the PPM2 and LMCSL-NU are mass conserving and shape preserving if necessary. The PPM2 scheme returns slightly better error scores, but the LMCSL-NU method is less damping as can be seen in Table II, where the LMCSL-NU method combined with the shape preserving filter, even after three revolutions, still retains the initial maximum value. In section 15 it is also shown that the LMCSL-NU method is significantly more multi-tracer efficient in real NWP applications.

7. Dynamic remapping

When using dynamic remapping, it is not possible to predict the exact amount of remappings that is needed at each time step. Therefore while running the model, the total number of dynamic remappings has been saved, so the difference between the methods can be appreciated. The dynamic remappings take additional time and will increase the computational load. It is also in the very nature of the dynamic remappings, that the amount will change from time step to time step. In an operational model it is unwanted, that the time used can not be determined precisely. It is therefore preferable if the total amount of remappings is approximately constant, so the time used per simulation is unchanged. Figure 11 illustrates the number of full

remappings, the individual dynamic remapping methods requires. The solid (red) curve is method I, the dot-dashed (blue) curve is method II, and the dotted (green) curve is the "ideal" method III. The x-axis is the time, the y-axis is the percentage of the total number of vertical intersects in the model, which is roughly 300,000. Method I requires almost no remappings. Within the first two hours it stabilizes at around 1.5%, and remains at that level after the initial increase. Method II, which has the same constraints, but instead remaps the Lagrangian levels to the allowed limit, shows a dramatic increase in the amount of remappings needed. It stabilizes, although not to quite as constant a level, after approximately twelve hours just below 15%. It is expected since the modified Lagrangian level is positioned exactly at the border, and it is very likely that an additional remapping is required at the following time step. Method III shows a gradual increase, somewhat slower than method I and method II, until it stabilizes at 2.5% after 20 hours.

Figure 11. Total percentage of remapped levels per time step. The solid (red) curve is dynamic remapping method I, the dot-dashed (blue) curve is dynamic remapping method II, and the dotted (green) curve is dynamic remapping method III.

The remappings are not distributed equally in the vertical. Figure 12 shows the distribution of the accumulated remappings after 24 hours. At that point all three methods has stabilized. Method I and II are very similar even though the amount of remappings is a factor ten different. In the upper part of the atmosphere, almost no remappings are needed, but in the middle layers the amount increases to a maximum between level 30 and 35. At the surface the remappings decrease. Method II reaches an equilibrium near the surface, where the number of remappings stabilizes. This is again because the levels that has been remapped, stays at the allowed limit. Method III has a tiny maximum close to the top, where some layers converge until they are remapped, and a large maxima close
to the surface where the layers are converging as well. In the
middle layers method III has almost no remappings since
the layers are allowed to move much more than with method
I and II (possibly several Eulerian levels).

8. **Vertical numerical diffusion**

Since the model is a fully operational NWP model, it is
difficult to assess the exact amount of unphysical numerical
vertical diffusion, as it cannot be separated from the real
diffusion. The following test has therefore been constructed
to quantify the vertical diffusion.

---

### Table I. Error norms for test case A

<table>
<thead>
<tr>
<th>Test</th>
<th>Time steps</th>
<th>( l_1 )</th>
<th>( l_2 )</th>
<th>( l_\infty )</th>
<th>( \Delta_{\text{mass}} )</th>
<th>( \phi_{\text{max}} )</th>
<th>( \phi_{\text{min}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL</td>
<td>100</td>
<td>0.1050</td>
<td>0.2004</td>
<td>0.3422</td>
<td>-1.3E-5</td>
<td>0.9427</td>
<td>0.0548</td>
</tr>
<tr>
<td>PPM-2</td>
<td>100</td>
<td>0.0736</td>
<td>0.1621</td>
<td>0.3170</td>
<td>0.0</td>
<td>0.9473</td>
<td>0.0514</td>
</tr>
<tr>
<td>PPM-2-SP</td>
<td>100</td>
<td>0.0654</td>
<td>0.1650</td>
<td>0.2972</td>
<td>0.0</td>
<td>0.9000</td>
<td>0.1000</td>
</tr>
<tr>
<td>LMCSL</td>
<td>100</td>
<td>0.1054</td>
<td>0.2009</td>
<td>0.3434</td>
<td>0.0</td>
<td>0.9427</td>
<td>0.0546</td>
</tr>
<tr>
<td>LMCSL-SP</td>
<td>100</td>
<td>0.0851</td>
<td>0.1958</td>
<td>0.3337</td>
<td>0.0</td>
<td>0.9000</td>
<td>0.1000</td>
</tr>
<tr>
<td>SL</td>
<td>300</td>
<td>0.1441</td>
<td>0.2384</td>
<td>0.3662</td>
<td>3.4E-5</td>
<td>0.9706</td>
<td>0.0536</td>
</tr>
<tr>
<td>PPM-2</td>
<td>300</td>
<td>0.0932</td>
<td>0.1860</td>
<td>0.3476</td>
<td>0.0</td>
<td>0.9433</td>
<td>0.0541</td>
</tr>
<tr>
<td>PPM-2-SP</td>
<td>300</td>
<td>0.0810</td>
<td>0.1881</td>
<td>0.3250</td>
<td>0.0</td>
<td>0.9000</td>
<td>0.1000</td>
</tr>
<tr>
<td>LMCSL</td>
<td>300</td>
<td>0.1445</td>
<td>0.2391</td>
<td>0.3674</td>
<td>0.0</td>
<td>0.9699</td>
<td>0.0541</td>
</tr>
<tr>
<td>LMCSL-SP</td>
<td>300</td>
<td>0.1140</td>
<td>0.2294</td>
<td>0.3603</td>
<td>0.0</td>
<td>0.9000</td>
<td>0.1000</td>
</tr>
</tbody>
</table>

### Table II. Error norms for test case B

<table>
<thead>
<tr>
<th>Test</th>
<th>Time steps</th>
<th>( l_1 )</th>
<th>( l_2 )</th>
<th>( l_\infty )</th>
<th>( \Delta_{\text{mass}} )</th>
<th>( \phi_{\text{max}} )</th>
<th>( \phi_{\text{min}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL</td>
<td>80</td>
<td>0.1041</td>
<td>0.2013</td>
<td>0.3356</td>
<td>4.0E-1</td>
<td>0.9483</td>
<td>0.0491</td>
</tr>
<tr>
<td>PPM-2</td>
<td>80</td>
<td>0.0903</td>
<td>0.1926</td>
<td>0.3902</td>
<td>0.0</td>
<td>0.9344</td>
<td>0.0623</td>
</tr>
<tr>
<td>PPM-2-SP</td>
<td>80</td>
<td>0.0949</td>
<td>0.2042</td>
<td>0.3857</td>
<td>0.0</td>
<td>0.8958</td>
<td>0.1000</td>
</tr>
<tr>
<td>LMCSL</td>
<td>80</td>
<td>0.1067</td>
<td>0.2062</td>
<td>0.3973</td>
<td>0.0</td>
<td>0.9485</td>
<td>0.0492</td>
</tr>
<tr>
<td>LMCSL-SP</td>
<td>80</td>
<td>0.0866</td>
<td>0.2018</td>
<td>0.3865</td>
<td>0.0</td>
<td>0.9000</td>
<td>0.1000</td>
</tr>
<tr>
<td>SL</td>
<td>240</td>
<td>0.1437</td>
<td>0.2391</td>
<td>0.3656</td>
<td>4.0E-1</td>
<td>0.9762</td>
<td>0.0527</td>
</tr>
<tr>
<td>PPM-2</td>
<td>240</td>
<td>0.1164</td>
<td>0.2231</td>
<td>0.4069</td>
<td>0.0</td>
<td>0.9347</td>
<td>0.0734</td>
</tr>
<tr>
<td>PPM-2-SP</td>
<td>240</td>
<td>0.1202</td>
<td>0.2281</td>
<td>0.3918</td>
<td>0.0</td>
<td>0.8808</td>
<td>0.1000</td>
</tr>
<tr>
<td>LMCSL</td>
<td>240</td>
<td>0.1435</td>
<td>0.2425</td>
<td>0.4120</td>
<td>0.0</td>
<td>0.9828</td>
<td>0.0511</td>
</tr>
<tr>
<td>LMCSL-SP</td>
<td>240</td>
<td>0.1145</td>
<td>0.2326</td>
<td>0.3999</td>
<td>0.0</td>
<td>0.9000</td>
<td>0.1000</td>
</tr>
</tbody>
</table>
At the first time step three layers in the model are initialized with a given value, the three layers are respectively in the top, middle, and bottom of the atmosphere. Here the layers $k = [5, 20, 35]$ have been chosen, and the mass is initialized at 100 kg in every cell. Figure 13 shows the mean of the sum of the mass present in each layer of the model, therefore giving us the vertical mass distribution at each time step. In the absence of diffusion, the mass would be constrained to the three initial layers.

The test has been run with three different settings - LMCSL, LMCSL-LL with dynamic remapping - method III, LMCSL-LL with fixed remapping every hour (6 steps). The duration is 24 hours of simulation corresponding to 144 steps of 600 seconds. As expected the traditional LMCSL scheme has the largest amount of vertical diffusion, but what is more interesting is the middle panel, where the effect of the fixed remapping can be seen. The effect is somewhat limited, which is surprising as the full remappings are only one sixth of the traditional. But since the vertical remapping is quite accurate when the levels only diverge slightly as in the traditional scheme, the remapping becomes less so when the levels diverge more. This could account for the reduced effect. The lower panel shows a drastic reduction of vertical diffusion, just as expected. The upper and middle level experiences almost no diffusion, and the surface layers has slightly less diffusion as well. The surface layers do introduce a strong mixing, which is directly related to the real parameterized turbulent mixing in the boundary layer, but without the additional numerical mixing.

Figure 14 shows the mass retained in the initial levels (5, 20, and 35), again all schemes show some vertical mixing in the upper and middle layers, depending on the method. In the lower layers the mixing is almost the same for all methods, which is fully consistent with the vertical mixing parameterized in the model.

### Figure 13
Vertical diffusion after 24 hours forecast. Upper panel shows the traditional LMCSL scheme, middle panel shows the LMCSL-LL with fixed remapping every hour, and the lower panel shows the LMCSL-LL with dynamic remapping 3.

### Figure 14
Vertical diffusion after 24 hours forecast. The lines indicate the mass retained in the emission levels. The dashed (green) lines are the traditional LMCSL scheme, the dot-dashed (black) lines are the LMCSL-LL with fixed remapping every hour, and the solid (red) lines are the LMCSL-LL with dynamic remapping 3.

### 9. Remapping efficiency
The increased amount of vertical interpolations (remappings) are the main factor in reducing the efficiency of the LMCSL-LL scheme. Therefore the new mass conserving
interpolation routine, LMCSL-NU, was created. Figure 15 shows the amount of CPU-time used in a full remapping of different number of chemical species. The solid line (red) is the LMCSL-NU method, the dot-dashed (blue) is the PPM2 method, and the dotted (green) line is the non-conserving and non-monotone Nevilles algorithm\textsuperscript{1}. It is evident that the PPM2-method is faster for a small number of fields, but already when four fields are remapped the new method becomes more efficient. The full Enviro-HIRLAM chemical scheme currently uses 36 chemical species. The traditional cubic interpolation, using Nevilles algorithm, has no multi-tracer efficiency and it cannot be used for mass conserving interpolations. But since the LMCSL-NU scheme uses the non-conserving interpolation weights, the same weights can be used for all interpolations, also enhancing consistency.

10. NWP forecast

A thorough validation calls for a long term comparison of forecast skill in simulations based on our new quasi-Lagrangian coordinate scheme with that in a reference operational model. Such validation is beyond the scope of the present paper, and instead, to illustrate that the new scheme performs realistically, we have compared meteorological fields in a single forecast. The reference model we compare with is HIRLAM (Undén et al. 2002) in a traditional NWP setup, which until recently has been used operationally at a number of weather forecast centers in Europe. We have chosen the validation period 23rd to the 26th of October 1994 at 12 UTC. This synoptic situation is from the ETEx experiment period, which originally was used for validating European dispersion models (Graziani et al. 1998) and Nodop et al. (1998).

As an example Figure 16 shows mean sea level pressure and temperature at 850 hPa in the verifying ERA-Interim re-analysis, and in 48 hour forecasts with the standard and the LMCSL-LL versions of HIRLAM valid on 25th October 1994 at 00 UTC. Similarly, Figure 17 shows the 6 hourly accumulated precipitation in the two versions of HIRLAM. In general LMCSL-LL is quite similar to the standard version of HIRLAM, and we therefore conclude that the new dynamical core of LMCSL-LL provide “plausible” forecasts. As expected there are, however, some differences. In particular it is worth mentioning the somewhat different distribution of precipitation with e.g. a stronger development of the frontal zone over eastern Europa in the standard HIRLAM, and in general, for this particular case, the standard HIRLAM shows stronger precipitation. No attempts have been made to verify the simulated precipitation against station data.

11. Conclusions

In this paper a new numerical scheme using a Lagrangian vertical coordinate has been introduced and tested against a more traditional quasi-Lagrangian numerical scheme with vertical remappings taking place every time step. Both schemes are based on the LMCSL method which conserves mass locally at very little additional cost. The Lagrangian coordinate scheme is implemented in an operational NWP model with the capability for online coupled chemistry. As part of the development a new vertical interpolation routine has been developed and tested using a traditional rotation test with a step function on a variable resolution.

The new scheme shows a decrease in unphysical numerical mixing when running in fully Lagrangian mode, where the levels are never remapped back to the Eulerian model levels. However, even a relatively small amount of full remappings, introduces a considerable amount of numerical diffusion in the vertical direction. The scheme is very multi-tracer efficient, as the additional prognostic variables, such as chemical species, only increases the overall computing cost by a minimum. The LMCSL-LL scheme is also shown to be very flexible, by enabling it to be run both in traditional mode (LTS = 0) or with many configurations for the remapping strategies including: fixed, dynamical, and combined remapping. The scheme produces realistic forecasts as compared to a traditional NWP model.

The new interpolation routine LMCSL-NU is shown to be very efficient and robust when used on very non-uniform grids, while still returning a shape preserving and mass conserving solution, with a very simple and local filter.

Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>NWP</td>
<td>Numerical weather prediction</td>
</tr>
<tr>
<td>ACT</td>
<td>Atmospheric chemical transport</td>
</tr>
<tr>
<td>SL</td>
<td>Semi-Lagrangian</td>
</tr>
<tr>
<td>SISL</td>
<td>Semi-implicit semi-Lagrangian</td>
</tr>
<tr>
<td>LMCSL</td>
<td>Locally mass conserving semi-Lagrangian</td>
</tr>
<tr>
<td>LMCSL-LL</td>
<td>LMCSL Lagrangian levels</td>
</tr>
<tr>
<td>LMCSL-NL</td>
<td>LMCSL non-uniform</td>
</tr>
<tr>
<td>LMCSL-M</td>
<td>LMCSL monotonic</td>
</tr>
<tr>
<td>LMCSL-MS</td>
<td>LMCSL simple monotonic</td>
</tr>
<tr>
<td>HIRLAM</td>
<td>High resolution limited area model</td>
</tr>
<tr>
<td>CISL</td>
<td>Cell integrated semi-Lagrangian</td>
</tr>
<tr>
<td>PPM2</td>
<td>Piecewise parabolic method</td>
</tr>
<tr>
<td>LTS</td>
<td>Lagrangian time step</td>
</tr>
<tr>
<td>ETEx</td>
<td>European Tracer Experiment</td>
</tr>
</tbody>
</table>

Acknowledgement

The present study was a part of the research of the Center for Energy, Environment and Health (CEEH), financed by The Danish Strategic Research Program on Sustainable Energy under contract no 09-061417.

\textsuperscript{1}Nevilles algorithm was originally used for vertical interpolations of the non-density variables, i.e. wind speed and temperature. It is of third order as the LMCSL-NU interpolation.
Figure 16. Forecasts of mean sea level pressure and 850 hPa temperature compared with verifying ERA40 reanalysis. From left to right: ERA-Interim re-analysis on 25th October 1994 00 UTC, 48 hour forecast with the standard HIRLAM, and 48 hour forecast with the LMCSL-LL (method III). The contour lines are the mean sea level pressure and the colored contours are the temperature at 850 mb.

Figure 17. Standard HIRLAM (left) and LMCSL-LL (right) forecasts of mean sea level pressure (same as Figure 16 and precipitation. The colored contours show 6 hourly accumulated precipitation up to 25th October 1994 00 UTC.


