Linear and Weakly Nonlinear Energetics of Global Nonhydrostatic Normal Modes

Carlos Frederico M. Raupp* and André Teruya
Department of Atmospheric Sciences
University of Sao Paulo, Sao Paulo/Brazil

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Introduction

- Conventional coarse resolution global circulation atmospheric models (AGCMs) neglect the vertical acceleration to prevent computational constraints related to stability of numerical solution in explicit numerical schemes (hydrostatic assumption) → Hydrostatic AGCMs;

- The increase of computer speed and memory, as well as the development of massively parallel computing techniques, has allowed the development of global nonhydrostatic modeling.

- Thus, with the increasing development of global nonhydrostatic AGCMs, it is important to understand the dynamics of these models in a theoretical point of view → For this purpose one needs to analyze the normal modes of global nonhydrostatic atmospheric models;

- Normal modes → small amplitude oscillations around a background state at rest and characterized by a stable stratification → eigensolutions of linearized PDEs;
References on Nonhydrostatic Normal Modes

- Linear theory of nonhydrostatic normal modes:
  
  (i) Kasahara and Qian (MWR 2000) ⇒ “Normal modes of a Global Nonhydrostatic Atmospheric Model”;
  
  
  (iii) Kasahara (J. Meteor. Soc. Japan, 2003) ⇒ “On the Nonhydrostatic Atmospheric Models with the Inclusion of the Horizontal Component of the Earth’s Angular Velocity” (non-traditional Coriolis terms);
  
  (iv) Kasahara (JAS 2003) ⇒ “The Roles of the Horizontal Component of the Earth’s Angular Velocity in Nonhydrostatic Linear Models”;
  
Introduction

Goal of this Study

(i) First we further analyze the energetics of the linear eigenmodes of the shallow global nonhydrostatic model presented by Kasahara and Qian (2000);

(ii) Then we extend the theory of global nonhydrostatic normal modes by accounting for the effect of nonlinearity.
Model and Governing Equations

**Model**: shallow nonhydrostatic fluid over a rotating sphere of radius \( a \);

Traditional Approximation: \( r = a + z \approx a \), where \( a = 6370\)Km (Earth’s radius) and \( z \) is the height above the earth’s surface; \( \partial / \partial r \approx \partial / \partial z \)

**Governing Equations**:

\[
\frac{Du}{Dt} - \left( f + \frac{u \tan \varphi}{a} \right) v = -\frac{1}{\rho a \cos \varphi} \frac{\partial p}{\partial \lambda} \tag{1a}
\]

\[
\frac{Dv}{Dt} + \left( f + \frac{u \tan \varphi}{a} \right) u = -\frac{1}{\rho a} \frac{\partial p}{\partial \varphi} \tag{1b}
\]

\[
\delta_H \frac{Dw}{Dt} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} \tag{1c}
\]

\[
\frac{D\rho}{Dt} + \rho \left( \nabla \cdot \vec{V} + \frac{\partial w}{\partial z} \right) = 0 \tag{1d}
\]

\[
\frac{Dp}{Dt} = \gamma RT \frac{D\rho}{Dt} \tag{1e}
\]

\[
p = \rho RT \tag{1f}
\]
From the governing equations (1), we have considered small (but not infinitesimal) amplitude perturbations around a resting, hydrostatic and isothermal background state:

\[ u = u_0 + u'; \quad v = v_0 + v'; \quad w = w_0 + w', \quad \text{with} \quad u_0 = v_0 = w_0 = 0; \]

\[ p = p_0(z) + p'; \quad \rho = \rho_0(z) + \rho'; \quad \text{with} \quad \frac{dp_0}{dz} = -\rho_0 g \quad \text{and} \quad T = T_0 + T', \quad \text{with} \quad T_0 = \text{const}; \quad (2) \]

Inserting (2) into (1) and retaining only the terms until second order in terms of perturbations:

\[ \frac{\partial u'}{\partial t} - f v' + \frac{1}{\rho_0 a \cos \phi} \frac{\partial p'}{\partial \lambda} = -\left[ \tilde{V}' \cdot \nabla u' + w' \frac{\partial u'}{\partial z} \right] + \frac{u' v'}{a} \tan \phi + \frac{\rho'}{\rho_0^2 a \cos \phi} \frac{\partial p'}{\partial \lambda} \quad (3a) \]

\[ \frac{\partial v'}{\partial t} + f u' + \frac{1}{\rho_0 a} \frac{\partial p'}{\partial \phi} = -\left[ \tilde{V}' \cdot \nabla v' + w' \frac{\partial v'}{\partial z} \right] - \frac{u'^2}{a} \tan \phi + \frac{\rho'}{\rho_0^2 a} \frac{\partial p'}{\partial \phi} \quad (3b) \]

\[ \frac{\partial w'}{\partial t} + \frac{1}{\rho_0} \left( \frac{\partial p'}{\partial z} + \frac{g}{C_s^2} (p' - \theta') \right) = -\left[ \tilde{V}' \cdot \nabla w' + w' \frac{\partial w'}{\partial z} \right] + \frac{\rho'}{\rho_0^2} \frac{\partial p'}{\partial z} + g \left( \frac{\rho'}{\rho_0} \right)^2 \quad (3c) \]

\[ \frac{1}{\rho_0 C_s^2} \left( \frac{\partial p'}{\partial t} - \rho_0 g w' \right) + \nabla \cdot \tilde{V}' + \frac{\partial w'}{\partial z} = -\frac{1}{C_s^2} \left[ \tilde{V}' \cdot \nabla p' + w' \frac{\partial p'}{\partial z} \right] - \rho' \left( \nabla \cdot \tilde{V}' + \frac{\partial w'}{\partial z} \right) + \frac{1}{C_s^2} \frac{T'}{T_0} \left[ \frac{\partial p'}{\partial t} - \rho_0 g w' \right] \quad (3d) \]

\[ \frac{\partial \theta'}{\partial t} + \rho_0 N^2 w' = \left[ \tilde{V}' \cdot \nabla \theta' + w' \frac{\partial \theta'}{\partial z} \right] + g \frac{T'}{C_s^2} \frac{T_0}{T_0} \left[ \frac{\partial p'}{\partial z} - \rho_0 g w' \right] \quad (3e) \]
Model and Governing Equations

Where: \( \theta' = \frac{g}{C_s^2} p' - g \rho' \) \quad (3f); \quad \frac{p'}{p_0} = \frac{T'}{T_0} + \frac{\rho'}{\rho_0} \quad (3g)

\[ N^2 = -g \left( \frac{1}{\rho_0} \frac{d\rho_0}{dz} + \frac{g}{C_s^2} \right) = \frac{\kappa g}{H} \]

\[ C_s^2 = \gamma RT_0 = \frac{gH}{1 - \kappa} \]

\[ H = \frac{RT_0}{g} \quad \kappa = \frac{R}{C_p} \]

Following Kasahara and Qian (2000) we have rescaled the perturbations according to:

\[
\begin{bmatrix}
u' \\
v' \\
w' \\
p' \\
\theta' \\
\rho'
\end{bmatrix} = \begin{bmatrix} u \rho_0 \frac{1}{2} \\ v \rho_0 \frac{1}{2} \\ w \rho_0 \frac{1}{2} \\ p \rho_0 \frac{1}{2} \\ \theta \rho_0 \frac{1}{2} \\ \rho \rho_0 \frac{1}{2} \end{bmatrix} \quad (4)
\]
Model and Governing Equations

- Substituting (4) into (3) we get:

\[
\frac{\partial u}{\partial t} - f v + \frac{1}{a \cos \varphi} \frac{\partial p}{\partial \lambda} = -\rho_0 \frac{1}{2} \left\{ [\vec{V} \cdot \nabla u + w L_z(u)] + \frac{uv}{a} \tan \varphi + \frac{\rho}{a \cos \varphi} \frac{\partial p'}{\partial \lambda} \right\}
\]  \hspace{1cm} (5a)

\[
\frac{\partial v}{\partial t} + f u + \frac{1}{a} \frac{\partial p}{\partial \varphi} = -\rho_0 \frac{1}{2} \left\{ [\vec{V} \cdot \nabla v + w L_z(v)] - \frac{u^2}{a} \tan \varphi + \frac{\rho}{a} \frac{\partial p'}{\partial \varphi} \right\}
\]  \hspace{1cm} (5b)

\[
\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} + \Gamma p - \theta = -\rho_0 \frac{1}{2} \left\{ [\vec{V} \cdot \nabla w + w L_z(w)] + \rho \frac{\partial p}{\partial z} + g \rho^2 \right\}
\]  \hspace{1cm} (5c)

\[
\frac{1}{C_s^2} \frac{\partial p}{\partial t} + \nabla \cdot \vec{V} + \frac{\partial w}{\partial z} - \Gamma w = -\rho_0 \frac{1}{2} \left\{ \frac{1}{C_s^2} \left[ \nabla \cdot \vec{V} p + w L_z^+(p) \right] - \rho \left( \nabla \cdot \vec{V} + L_z^-(w) \right) + \frac{1}{C_s^2} \left( \frac{p}{RT_0} - \rho \right) \left( \frac{\partial p}{\partial t} - gw \right) \right\}
\]  \hspace{1cm} (5d)

\[
\frac{1}{N^2} \frac{\partial \theta}{\partial t} + w = -\rho_0 \frac{1}{2} \left\{ \frac{1}{N^2} \left[ \nabla \cdot \vec{V} \theta + w L_z^+(\theta) \right] + \frac{g}{N^2 C_s^2} \left( \frac{p}{RT_0} - \rho \right) \left( \frac{\partial p}{\partial z} - gw \right) \right\}
\]  \hspace{1cm} (5e)

Where,

\[
L_z^+(\ ) = \frac{\partial}{\partial z} + \frac{1}{2 \rho_0} \frac{d \rho_0}{dz} = \frac{\partial}{\partial z} - \frac{1}{2H}
\]

\[
L_z^-(\ ) = \frac{\partial}{\partial z} - \frac{1}{2 \rho_0} \frac{d \rho_0}{dz} = \frac{\partial}{\partial z} + \frac{1}{2H}
\]

\[
\Gamma = \frac{1}{2 \rho_0} \frac{d \rho_0}{dz} + \frac{g}{C_s^2} = \frac{1 - 2 \kappa}{2H}
\]
If the second-order nonlinear terms are disregarded, equations (5) become:

\[
\frac{\partial u}{\partial t} - f v + \frac{1}{a \cos \varphi} \frac{\partial p}{\partial \lambda} = 0 \tag{6a}
\]

\[
\frac{\partial v}{\partial t} + f u + \frac{1}{a} \frac{\partial p}{\partial \varphi} = 0 \tag{6b}
\]

\[
\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} + \Gamma p - \theta = 0 \tag{6c}
\]

\[
\frac{1}{C_s^2} \frac{\partial p}{\partial t} + \nabla \cdot \vec{V} + \frac{\partial w}{\partial z} - \Gamma w = 0 \tag{6d}
\]

\[
\frac{1}{N^2} \frac{\partial \theta}{\partial t} + w = 0 \tag{6e}
\]

The eigensolutions of (6) were determined by Kasahara and Qian (2000) for the following boundary conditions:

(i) \( w = 0 \) at \( z = 0 \) and at \( z = z_T \) \( \tag{7a} \)
(ii) Periodic solutions in longitude \( \tag{7b} \)
(iii) Regularity at the poles \( \tag{7c} \)
The eigensolutions of (6) with boundary conditions (7) are given by:

\[
\begin{bmatrix}
    u \\
    v \\
    p \\
    w \\
    \theta
\end{bmatrix}
= 
\begin{bmatrix}
    U(\varphi)\xi(z) \\
    iV(\varphi)\xi(z) \\
    P(\varphi)\xi(z) \\
    iP(\varphi)\eta(z) \\
    P(\varphi)\Theta(z)
\end{bmatrix} 
\cdot e^{is\lambda - i\sigma t}
\]  

(8)

where:

\[
s\xi\left(\frac{1}{gH_e} - \frac{1}{C_s^2}\right) + \left(\frac{d\eta}{dz} - \Gamma \eta\right) = 0
\]

\[-\sigma\Theta + N^2\eta = 0
\]

\[
s\eta + \left(\frac{d\xi}{dz} + \Gamma \xi\right) - \Theta = 0
\]

- \sigma U - f V + \frac{sP}{a \cos \varphi} = 0

\[
\sigma V + f U + \frac{1}{a} \frac{dP}{d\varphi} = 0
\]

\[
\frac{1}{a \cos \varphi} \left[ s U + \frac{d}{d\varphi} \left( V \cos \varphi \right) \right] = \frac{\sigma P}{gH_e}
\]

Vertical structure equations

horizontal structure equations

(Laplace’s tidal equations)
The vertical structure equations can be written in terms of $\eta$ as follows:

$$\frac{d^2 \eta}{dz^2} + \left( \lambda - \Gamma^2 \right) \eta = 0 \quad ; \quad \text{with BCs: } \eta = 0 \text{ at } z = 0 \text{ and at } z = z_T \ ;$$

$$\lambda = \left( \frac{1}{gH_e} - \frac{1}{C_s^2} \right) \left( N^2 - \delta_H \sigma^2 \right)$$

The solution is given by:

$$\eta(z) = A_k \sin \left( \frac{k \pi}{z_T} \right), \ k = 1, 2, 3 \ldots \ , \text{ provided that}$$

$$\lambda_k = \left( \frac{k \pi}{z_T} \right)^2 + \Gamma^2 \quad \text{Eigenvalues of the vertical structure equations}$$

Separation constant:

$$H_e = \frac{C_s^2}{g} \left( 1 + \frac{\lambda_k C_s^2}{N^2 - \delta_H \sigma^2} \right)^{-1}$$

$$\sigma = F(s,l,H_e) \quad \text{Eigenvalues of the Laplace’s tidal equations}$$
Eigenmodes of the Linear Problem

Different oscillation regimes: (I) $H_e > \frac{C_S^2}{g} \rightarrow \sigma^2 > N^2 \rightarrow$ inertio-acoustic modes;

(ii) $H_e < \frac{C_S^2}{g} \rightarrow \sigma^2 < N^2 \rightarrow$ inertio-gravity modes;

Dispersion curves for acoustic and gravity modes for $l = 0$, $k = 1$, and symmetric about the equator.
Eigenmodes of the Linear Problem

- Different oscillation regimes: (I) $H_e > \frac{C_s^2}{g} \rightarrow \sigma^2 > N^2 \rightarrow$ inertio-acoustic modes;
  \[
  \text{(ii) } H_e < \frac{C_s^2}{g} \rightarrow \sigma^2 < N^2 \rightarrow \text{inertio-gravity modes;}
  \]

Equivalent heights $H_e$ for acoustic and gravity modes for $l = 0, k = 1$, and symmetric about the equator.
Energetics of Normal modes

Kasahara and Qian (2000) have demonstrated the orthogonality condition for the eigenmodes:

\[
\left(i \sigma_j - i \sigma_k\right) \int_0^\pi \int_0^\pi \int_0^\pi \left( u_j u_k^* + v_j v_k^* + \delta_H w_j w_k^* \right) + \frac{p_j p_k^*}{C_S^2} + \frac{\theta_j \theta_k^*}{N^2} a^2 \cos \varphi \, d\varphi \, d\lambda \, dz = 0
\]

For the case \( j = k \) we have the total energy of the \( j \)-th eigenmode of the system:

\[
ET_j = \int_0^\pi \int_0^\pi \int_0^\pi \left[ K_j + E_j + A_j \right] a^2 \cos \varphi \, d\varphi \, d\lambda \, dz > 0
\]

Where:

\[
K_j = \frac{1}{2} \int_0^\pi \int_0^\pi \int_0^\pi \left[ u_j^2 + v_j^2 + \delta_H w_j^2 \right] a^2 \cos \varphi \, d\varphi \, d\lambda \, dz \quad \text{Kinetic energy}
\]

\[
E_j = \frac{1}{2} \int_0^\pi \int_0^\pi \int_0^\pi \frac{p_j^2}{C_S^2} a^2 \cos \varphi \, d\varphi \, d\lambda \, dz \quad \text{elastic energy}
\]

\[
A_j = \frac{1}{2} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\theta_j^2}{N^2} a^2 \cos \varphi \, d\varphi \, d\lambda \, dz \quad \text{Termobaric energy}
\]
Energetics of eastward inertio-acoustic modes with meridional index $l = 0$ and $k = 1$. 
Energetics of eastward inertio-acoustic modes with meridional index $l = 0$ and $k = 1$. Each energy type is normalized by total energy.
Energetics of normal modes

Energetics of eastward inertio-gravity modes with meridional index $l = 0$ and $k = 1$. 
Energetics of eastward inertio-gravity modes with meridional index $l = 0$ and $k = 1$. Each energy type is normalized by total energy.
Resonant Nonlinear Interactions of Global Nonhydrostatic Modes

\[ \frac{\partial u}{\partial t} - f v + \frac{1}{a \cos \varphi} \frac{\partial p}{\partial \lambda} = -\rho_0^{\frac{1}{2}} \left\{ \left[ \mathbf{V} \cdot \nabla u + wL_z^-(u) \right] + \frac{uv}{a} \tan \varphi + \frac{\rho}{a \cos \varphi} \frac{\partial p'}{\partial \lambda} \right\} \] (5a)

\[ \frac{\partial v}{\partial t} + f u + \frac{1}{a} \frac{\partial p}{\partial \varphi} = -\rho_0^{\frac{1}{2}} \left\{ \left[ \mathbf{V} \cdot \nabla v + wL_z^-(v) \right] - \frac{u^2}{a} \tan \varphi + \frac{\rho}{a} \frac{\partial p'}{\partial \varphi} \right\} \] (5b)

\[ \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} + \Gamma p - \theta = -\rho_0^{\frac{1}{2}} \left\{ \left[ \mathbf{V} \cdot \nabla w + wL_z^-(w) \right] + \rho \frac{\partial p}{\partial z} + g \rho^2 \right\} \] (5c)

\[ \frac{1}{C_s^2} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{V} + \frac{\partial w}{\partial z} - \Gamma w = \rho_0^{\frac{1}{2}} \left\{ -\frac{1}{C_s^2} \left[ \mathbf{V} \cdot \nabla p + wL_z^+ (p) \right] - \rho \left( \nabla \cdot \mathbf{V} + L_z^+ (w) \right) + \frac{1}{C_s^2} \left( \frac{p}{RT_0} - \rho \right) \left( \frac{\partial p}{\partial t} - gw \right) \right\} \] (5d)

\[ \frac{1}{N^2} \frac{\partial \theta}{\partial t} + w = \rho_0^{\frac{1}{2}} \left\{ -\frac{1}{N^2} \left[ \mathbf{V} \cdot \nabla \theta + wL_z^+ (\theta) \right] + \frac{g}{N^2 C_s^2} \left( \frac{p}{RT_0} - \rho \right) \left( \frac{\partial p}{\partial t} - gw \right) \right\} \] (5e)

Where,

\[ L_z^+(\ ) = \frac{\partial}{\partial z} + \frac{1}{2 \rho_0} \frac{d \rho_0}{dz} = \frac{\partial}{\partial z} - \frac{1}{2H} \]

\[ L_z^-(\ ) = \frac{\partial}{\partial z} - \frac{1}{2 \rho_0} \frac{d \rho_0}{dz} = \frac{\partial}{\partial z} + \frac{1}{2H} \]

\[ \Gamma = \frac{1}{2 \rho_0} \frac{d \rho_0}{dz} + \frac{g}{C_s^2} = \frac{1 - 2 \kappa}{2H} \]
Resonant Interactions of nonhydrostatic nonnormal modes: general case

**Ansatz** ⇒ Solution with **three** modes:

\[
\begin{bmatrix}
\text{u} \\
\text{v} \\
\text{w} \\
\text{p} \\
\text{θ} \\
\text{ρ}
\end{bmatrix}
(\lambda, \varphi, z, t) = A_1(t) \begin{bmatrix} U_1(\varphi, z) \\ iV_1(\varphi, z) \\ iW_1(\varphi, z) \\ P_1(\varphi, z) \\ \theta_1(\varphi, z) \\ \rho_1(\varphi, z) \end{bmatrix} e^{is_1\lambda - i\sigma_1 t} + A_2(t) \begin{bmatrix} U_2(\varphi, z) \\ iV_2(\varphi, z) \\ iW_2(\varphi, z) \\ P_2(\varphi, z) \\ \theta_2(\varphi, z) \\ \rho_2(\varphi, z) \end{bmatrix} e^{is_2\lambda - i\sigma_2 t} + A_3(t) \begin{bmatrix} U_3(\varphi, z) \\ iV_3(\varphi, z) \\ iW_3(\varphi, z) \\ P_3(\varphi, z) \\ \theta_3(\varphi, z) \\ \rho_3(\varphi, z) \end{bmatrix} e^{is_3\lambda - i\sigma_3 t} + C.C
\]

With the following resonance relations satisfied:

- \( k_3 = k_1 + k_2 \) (not excluding)
- \( s_1 = s_2 + s_3 \)
- \( \sigma_1 = \sigma_2 + \sigma_3 \)

condition for meridional structures satisfied

\[
I_z = \int_0^{z_T} \rho_0 \frac{1}{2} \cos \left( k_1 \pm k_2 \pm k_3 \frac{\pi z}{z_T} \right) dz
\]

Nonlinear resonant triad interaction conditions
Substituting the ansatz into the PDEs (5) we get:

\[
ET_1 \frac{dA_1}{dt} = i \alpha_{12}^{23} A_2 A_3
\]

\[
ET_2 \frac{dA_2}{dt} = i \alpha_{23}^{13} A_1 A_3^* \]

\[
ET_3 \frac{dA_3}{dt} = i \alpha_{31}^{12} A_1 A_2^* \]

\(\alpha_1^{23}, \alpha_2^{13}, \alpha_3^{12} \Rightarrow \text{Nonlinear coupling constants;}\)
Resonant Interactions of nonhydrostaic nonrmal modes: general case

\[ \alpha_{1}^{23} = \int_{0}^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{z_{T}} \left[ N_{u}^{(2,3)} u_{1}^{*} + N_{v}^{(2,3)} v_{1}^{*} + N_{w}^{(2,3)} w_{1}^{*} + N_{u}^{(2,3)} u_{1}^{*} + N_{p}^{(2,3)} p_{1}^{*} + N_{\theta}^{(2,3)} \theta_{1}^{*} \right] a^{2} \cos \phi d \phi \rho_{0}^{2} dz \]

\[ \alpha_{2}^{13} = \int_{0}^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{z_{T}} \left[ N_{u}^{(1,3)} u_{2}^{*} + N_{v}^{(1,3)} v_{2}^{*} + N_{w}^{(1,3)} w_{2}^{*} + N_{u}^{(1,3)} u_{2}^{*} + N_{p}^{(1,3)} p_{2}^{*} + N_{\theta}^{(1,3)} \theta_{2}^{*} \right] a^{2} \cos \phi d \phi \rho_{0}^{2} dz \]

\[ \alpha_{3}^{12} = \int_{0}^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{z_{T}} \left[ N_{u}^{(1,2)} u_{3}^{*} + N_{v}^{(1,2)} v_{3}^{*} + N_{w}^{(1,2)} w_{3}^{*} + N_{u}^{(1,2)} u_{3}^{*} + N_{p}^{(1,2)} p_{3}^{*} + N_{\theta}^{(1,2)} \theta_{3}^{*} \right] a^{2} \cos \phi d \phi \rho_{0}^{2} dz \]
Resonant Interactions of nonhydrostaic nonrmal modes: general case

\[
N^{(2,3)}_u = \left[ u_2 \frac{is_3 u_3}{a \cos \varphi} + \frac{v_2}{a} \frac{\partial u_3}{\partial \varphi} + w_2 L_z^-(u_3) \right] + \frac{u_2 v_3}{a} \tan \varphi + \frac{\rho_2}{a \cos \varphi} is_3 p_3 + CP
\]

\[
N^{(2,3)}_v = \left[ u_2 \frac{is_3 v_3}{a \cos \varphi} + \frac{v_2}{a} \frac{\partial v_3}{\partial \varphi} + w_2 L_z^-(v_3) \right] - \frac{u_2 u_3}{a} \tan \varphi + \frac{\rho_2}{a} \frac{\partial p_3}{\partial \varphi} + CP
\]

\[
N^{(2,3)}_w = -\delta_H \left[ \frac{u_2}{a \cos \varphi} is_3 w_3 + \frac{v_2}{a} \frac{\partial w_3}{\partial \varphi} + w_2 L_z^-(w_3) \right] + \rho_2 L_z^+(p_3) + g\rho_2 \rho_3 + CP
\]

\[
N^{(2,3)}_p = -\frac{1}{C_s^2} \left[ \frac{u_2}{a \cos \varphi} is_3 p_3 + \frac{v_2}{a} \frac{\partial p_3}{\partial \varphi} + w_2 L_z^+(p_3) \right] - \rho_2 \left[ \frac{1}{a \cos \varphi} \left( is_3 u_3 + \frac{\partial (v_3 \cos \varphi)}{\partial \varphi} \right) + L_z^-(w_3) \right]
\]

\[+ \frac{1}{C_s^2} \left( \frac{p_2}{RT_0} - \rho_2 \right)(-i\sigma_3 p_3 - gw_3) + CP
\]

\[
N^{(2,3)}_\theta = -\frac{1}{N^2} \left[ \frac{u_2}{a \cos \varphi} is_3 \theta_3 + \frac{v_2}{a} \frac{\partial \theta_3}{\partial \varphi} + w_2 L_z^+(\theta_3) \right] + \frac{g}{C_s^2 N^2} \left( \frac{p_2}{RT_0} - \rho_2 \right)(-i\sigma_3 p_3 - gw_3) + CP
\]
Resonant Interactions of nonhydrosticaic nonnormal modes

From the complex amplitude equations it is easy to get the energy equations:

\[ ET_1 \frac{d|A_1|^2}{dt} = -\alpha_1^{23} \text{Im}(A_1 A_2^* A_3^*) \]

\[ ET_2 \frac{d|A_2|^2}{dt} = \alpha_2^{13} \text{Im}(A_1 A_2^* A_3^*) \]

\[ ET_3 \frac{d|A_3|^2}{dt} = \alpha_3^{12} \text{Im}(A_1 A_2^* A_3^*) \]

Condition for total energy to be conserved within a resonant triad interaction is:

\[ -\alpha_1^{23} + \alpha_2^{13} + \alpha_3^{12} = 0 \]

Mode 1 \(\Rightarrow\) unstable mode of the triad.
Resonant Interactions between Acoustic and Gravity Modes

- Analytical solutions of the conservative triad equations, assuming that

\[
\left| \alpha_3^{12} \right| < \left| \alpha_2^{13} \right| < \left| \alpha_1^{23} \right| \]

and the amplitude of mode 1 is zero initially:

\[
ET_1 |A_1(t)|^2 = |A_2(0)|^2 \left( \frac{\alpha_1^{23}}{\alpha_2^{13}} \right) sn^2 \left( \frac{u}{m} \right)
\]

\[
ET_2 |A_2(t)|^2 = |A_2(0)|^2 \ cn^2 \left( \frac{u}{m} \right)
\]

\[
ET_3 |A_3(t)|^2 = |A_3(0)|^2 \ dn^2 \left( \frac{u}{m} \right)
\]

Where \( sn, cn \) and \( dn \) are the Jacobian Elliptic functions, with argument \( u \) and parameter \( m \) given by

\[
u = \left| A_3(0) \right| \left| \alpha_1^{23} \alpha_2^{13} \right| t
\]

\[
m = \frac{\alpha_3^{12}}{\alpha_2^{13}} \left( \frac{|A_2(0)|}{|A_3(0)|} \right)^2
\]
Resonant Interactions between Acoustic and Gravity Modes

- Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:

Determination of a resonant triad involving a long inertio-acoustic, a short acoustic mode and a short gravity mode. The acoustic modes have $k = 1$ vertical structure, while the gravity mode has a $k = 2$ vertical structure.

![Graph showing frequency versus wavenumber for acoustic and gravity modes.](image)

Determination of a resonant triad involving a long inertio-acoustic, a short acoustic mode and a short gravity mode. The acoustic modes have $k = 1$ vertical structure, while the gravity mode has a $k = 2$ vertical structure.
Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:

Mode 1: **unstable (pump) mode**
Acoustic mode with $k = 1$, $s = 476$, $l = 0$ (first symmetric mode)

Mode 2:
Acoustic mode with $k = 1$, $s = 1$, $l = 0$ (first symmetric mode)

Mode 3:
Gravity mode with $k = 2$, $s = 475$, $l = 0$ (first symmetric mode)

<table>
<thead>
<tr>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>$\sigma_1$ (cHz)</th>
<th>$\sigma_2$ (cHz)</th>
<th>$\sigma_3$ (cHz)</th>
<th>$\alpha_{12}^{23}$</th>
<th>$\alpha_{23}^{12}$</th>
<th>$\alpha_{32}^{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,476,0,A)</td>
<td>(1,1,0,A)</td>
<td>(2,475,0,G)</td>
<td>6.293</td>
<td>5.888</td>
<td>0.407</td>
<td>1.3x$10^{10}$</td>
<td>1.3x$10^{20}$</td>
<td>7x$10^{6}$</td>
</tr>
</tbody>
</table>
Resonant Interactions between Acoustic and Gravity Modes

Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:

\[ 0 < m \ll 1 \]
Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:

\[ 0 < m < 1 \]
Resonant Interactions between Acoustic and Gravity Modes

- Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:

\[ 0 \ll m < 1 \]
Vertical velocity at $\varphi = 10^\circ S$ and $z = 9\text{ Km}$; Short acoustic mode activity.

$0 \ll m < 1$

Zonal velocity at $\varphi = 0^\circ$ and $z = 4.5\text{ Km}$

Short gravity mode activity.
Summary and Remarks

- Here we have investigated the possibility of resonant interactions involving inertio-acoustic and inertio-gravity modes in a shallow-nonhydrostatic global atmospheric model (weakly nonlinear extension of Kasahara and Qian (2000)).

- For the internal modes (rigid lid boundary condition), we found that the only possibility for such resonances is that one gravity mode interacts with two acoustic modes (similar to Rossby-gravity-gravity interaction in the hydrostatic dynamics);

- This kind of resonant interaction can potentially yield vacillations in the dynamical fields with periods varying from a daily (and intra-diurnal) time-scale up to almost a month long, depending on the way in which the initial energy is distributed on the triad components;

- Acoustic modes are usually filtered out from numerical models to avoid computational constraints associated with explicit numerical schemes, even in nonhydrostatic models;
Next Steps of the Project

- To investigate the possibility of resonant interactions for the limiting case of vertical modes where \( z_T \rightarrow \infty \).

- To study the possibility of long-short wave interactions.

- To investigate the dynamics of these resonant interactions with the inclusion of diabatic effects;