3D Spectral Energetics Analysis and Rossby Wave Saturation Theory

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Lorenz Energy Box Diagram

Energy accumulates at zonal wave 0

Energy accumulates at vertical wave 0

(Kung and Tanaka 1983, JAS)
Lorenz cycle, Saltzman cycle

(Saltzman 1957; 1970) \( p = \sum_{n=-\infty}^{\infty} p_n \exp(inx) \)

\[
\frac{\partial K_z}{\partial t} = \sum_{n=1}^{N} M(n) + C(0) - D(0),
\]

\[
\frac{\partial K(n)}{\partial t} = -M(n) + L(n) + C(n) - D(n), \quad n = 1, 2, 3, \ldots
\]

\[
\frac{\partial P_z}{\partial t} = -\sum_{n=1}^{N} R(n) - C(0) + G(0),
\]

\[
\frac{\partial P(n)}{\partial t} = R(n) + S(n) - C(n) + G(n), \quad n = 1, 2, 3, \ldots
\]
G: Generation of P(n)
P: Available potential energy
R: zonal-wave interaction of P(n)
S: wave-wave interaction of P(n)
C: Baroclinic conversion from P(n) to K(n)
K: Kinetic energy
M: zonal-wave interaction of K(n)
L: wave-wave interaction of K(n)
D: Dissipation of K(n)

(Saltzman 1957 & 1970)
(Kung and Tanaka 1983 & 1984)
Data

- **JRA-25**
  - $2.5^\circ \times 2.5^\circ$, 23 levels (1000 - 0.4 hPa)
- **NCEP/NCAR reanalysis**
  - $2.5^\circ \times 2.5^\circ$, 17 levels (1000 - 10 hPa)
- **ERA-40**
  - $2.5^\circ \times 2.5^\circ$, 23 levels (1000 - 1 hPa)
- **1990/91 DJF (3 Month)**
  - $u$, $v$, $T$, $q$
Available Potential Energy $P(n)$
Kinetic energy $K(n)$

- NCEP up to $n=35$
- ERA up to $n=60$

<table>
<thead>
<tr>
<th></th>
<th>NCEP</th>
<th>JRA</th>
<th>ERA</th>
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<tr>
<td>$P_E$</td>
<td>64</td>
<td>67</td>
<td>70</td>
</tr>
<tr>
<td>$K_E$</td>
<td>77</td>
<td>82</td>
<td>84</td>
</tr>
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</table>
Wave-wave interactions (S, L)

\[ \sum_{i=1}^{n} S(i) \]

\[ \sum_{i=1}^{n} L(i) \]

Energy Flux

Up-scale cascade

Down-scale cascade

n=2\sim7

n=6\sim11
2D Spectral model

- 1D: Expansion in Fourier harmonics
- 2D: Expansion in spherical harmonics

\[ p = \sum_{n=-\infty}^{\infty} p_n \exp(inx) \]

\[ \frac{\partial p}{\partial x} = \sum_{n=-\infty}^{\infty} in p_n \exp(inx) \]

\[ Y_l^n (\lambda, \theta) = P_l^n (\theta) \exp(in\lambda) \]

\[ p(\lambda, \theta) = \sum_{n=-N}^{N} \sum_{l=|n|}^{L} p_{nl} Y_l^n (\lambda, \theta) \]
3D Spectral model

• Vertical normal mode

• Horizontal normal mode: Hough harmonics

Expansion in 3D Normal Mode Functions

\[ \Pi_{nlm}(\lambda, \theta, \sigma) = \Theta_{nlm}(\theta) G_m(\sigma) \exp(in\lambda) \]

\[ U(\lambda, \theta, \sigma) = \sum_{n=-N}^{N} \sum_{l=0}^{L} \sum_{m=0}^{M} w_{nlm} X_m \Pi_{nlm}(\lambda, \theta, \sigma) \]

\[ \frac{dw_i}{d\tau} = -i\sigma_i w_i - i \sum_{jk} r_{ijk} w_j w_k + f_i, \quad i = 1, 2, 3, \ldots \]

\[ E_i = \frac{1}{2} p_s h_m |w_i|^2, \quad w_{nlm} \rightarrow w_i \]
Vertical energy spectrum

Vertical modes

\[ G_0(\sigma) = C_1 \sigma^n + C_2 \sigma^{n_2} \]

\[ G_m(\sigma) = \sigma^{-\frac{1}{2}} \left( C_1 \sin(\mu \ln \sigma) + C_2 \cos(\mu \ln \sigma) \right) \]

Barotropic and baroclinic modes

\[ \lambda_m = \frac{R\gamma}{gh_m} \]

\[ \mu = \sqrt{\lambda_m - \frac{1}{4}} \]

Terasaki and Tanaka (2007)
Barotropization by baroclinic instability

Barotropic mode

Energy flux of Kinetic energy

(Terasaki and Tanaka 2007)
Spherical harmonics (n=0)  

Hough harmonics  

Kelvin mode  
Mixed Rossby-gravity mode
Fig. 2. Energy distributions in the wavenumber domain. $K$: kinetic energy, $A$: available potential energy, $K_v$: $v$-component of $K$.

Fig. 3. Eddy energy distributions in the vertical mode domain.

Tanaka (1985)
Meridional Energy Spectrum

Tanaka (1985)
Energy spectrum in the 3D wavenumber space

\[ c = -\frac{\beta}{n^2 + l^2 + m^2} = -\frac{\beta}{k^2} \]

\( n, l, m \): zonal, meridional and vertical waves

\( k \): total wave \( c = \sigma / n \)

\[ \frac{dw_i}{d\tau} = -i\sigma_i w_i - i \sum_{jk} r_{ijk} w_j w_k + f_i \]

Use \( c \) for the scale in place of 3D wavenumber
Observed energy spectrum in \textit{c-domain}

- **Frequency domain**
  - Tanaka and Kasahara (1992)
  - Tanaka (1985)

- **Phase speed domain**
  - Shigehisa mode
  - $n=0$

Frequency domain

Tanaka and Kimura (1996)

Vertical mode
m=0

Vertical mode
m=1

Vertical mode
m=2

Vertical mode
m=3
We can construct a barotropic model with baroclinic instability.
Primitive Equation Model

\[ M \frac{\partial U}{\partial t} + LU = N + F, \]  \hspace{1cm} (1)

where

\[ U = (u, v, \phi')^T, \]  \hspace{1cm} (2)

\[ M = \text{diag}(1, 1, -\frac{\partial}{\partial p} R \frac{\partial}{\partial p} R), \]  \hspace{1cm} (3)

\[ L = \begin{pmatrix}
0 & -2\Omega \sin\theta & \frac{1}{a \cos\theta} \frac{\partial}{\partial\lambda} \\
2\Omega \sin\theta & 0 & \frac{1}{a \cos\theta} \frac{\partial}{\partial\theta} \\
\frac{1}{a \cos\theta} \frac{\partial}{\partial\lambda} & \frac{1}{a \cos\theta} \frac{\partial(\cos\theta)}{\partial\theta} & 0
\end{pmatrix}, \]  \hspace{1cm} (4)

\[ N = \begin{pmatrix}
-V \cdot \nabla u - \omega \frac{\partial u}{\partial p} + \frac{\tan\theta}{a} uu \\
-V \cdot \nabla v - \omega \frac{\partial v}{\partial p} - \frac{\tan\theta}{a} vv \\
\frac{\partial}{\partial p} \left( \frac{p^2}{R} V \cdot \nabla \frac{\partial \phi}{\partial p} + \omega p \left( \frac{\partial}{\partial p} \frac{Q}{C_p} \right) \right)
\end{pmatrix}, \]  \hspace{1cm} (5)

\[ F = (F_u, F_v, \frac{\partial}{\partial p} \left( \frac{pQ}{C_p \gamma} \right))^T. \]  \hspace{1cm} (6)
Numerical simulations of blocking and AO

Barotropic Component of the Atmosphere

- **Vertical Transform**

\[
(u, v, \phi')^T_0 = \frac{1}{p_s} \int_0^{p_s} (u, v, \phi')^T G_0 dp
\]  

- **Barotropic Model**

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\vec{v} \cdot \nabla u + fv - \frac{\partial \phi}{\partial x} + F_x \\
\frac{\partial v}{\partial t} &= -\vec{v} \cdot \nabla v - fu - \frac{\partial \phi}{\partial y} + F_y \\
\frac{\partial \phi}{\partial t} &= -\vec{v} \cdot \nabla \phi - \tilde{\phi} \nabla \cdot \vec{v} + F_z
\end{align*}
\]  

- **3-D Spectral Transform**

\[
U(\lambda, \theta, p, t) = \sum_{nlm} w_{nlm}(t) X_m \Pi_{nlm}(\lambda, \theta, p),
\]

\[
w_{nlm}(t) = \langle U(\lambda, \theta, p, t), X_m^{-1} \Pi_{nlm} \rangle
\]

where \(U(\lambda, \theta, p, t) = (u, v, \phi')^T\), \(w_{nlm}(t)\) is the spectral expansion coefficient, \(X_m = \text{diag}(c_m, c_m, c_m^2)\), and \(\Pi_{nlm}\) is the 3-D NMF.

Barotropic S-Model
(Tanaka 2003, JAS)
NCEP/NCAR

Barotropic Height
DIF mean for 1950-2000

Barotropic S-Model

Geopotential Height
Perpetual January (50 Years)

Perpetual January

Barotropic Energy (S-Model)
Arctic Oscillation
Barotropic height (EOF-1)

NCEP/NCAR
Barotropic Component of Geopotential Height
EOF-1 AO (5.7%)

Barotropic S-Model
Barotropic Height
EOF-1 (16%)
Singular Eigenmode Theory of AO

SVD Analysis
(Diffusion only)

Gain Function

EVP-2  EVP-1

Eddy Feedback Effect

52 Days  20 Days

Eigenmode

EOF-1

Tanaka and Matsueda (2005)
Blocking in the model

500 hPa Height
JMA GPV 97031412+00

Geopotential Height
Run-02 Day 955
Blocking formation by Rossby wave breaking

(Tanaka and Watarai 1999)
Rossby wave breaking and saturation

Time Series

Basic state of n=0

Eddies

Total Energy (Jm^-2)

0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200

Time in Day

10^0

10^1

10^2

10^3

10^4

10^5

10^6

10^7

breaking saturation

Potential Vorticity Wave-6 Model

Day 48

Day 50

Day 52

Day 54

Day 100

Day 101

Rossby Wave

q=const.

Barotropic Instability

Tanaka and Watarai (1999)

Mudrick (1974)
Zonalization

Rossby wave breaking for n=6

Growthrate $\times$ 1.7
3D energy spectrum

By 3D normal mode expansion

\[
\frac{dw_i}{d\tau} = -i\sigma_i w_i - i\sum_{jk} r_{ijk} w_j w_k + f_i
\]

\[
E_i = \frac{1}{2} p_s h_0 |w_i|^2
\]

\[
c_i = \sigma_i / n \quad \text{Phase speed}
\]

\[E = mc^2 \quad ?\]

Tanaka et al. (2004 GRL)
Saturation theory in gravity waves

Saturation spectrum

-3/5 power

Breaking gravity waves

Breaking condition

\[ \frac{\partial \theta}{\partial z} < 0 \]

Fritts (1984)
Saturation theory in Rossby waves

\[ \frac{\partial q}{\partial y} < 0, \quad q = \nabla^2 \psi + f \]

\[ \frac{\partial}{\partial y} (\nabla^2 \psi + f) = -\nabla^2 u + \beta < 0 \]

\[ u < -\frac{\beta}{n^2 + l^2} = c \]

Tanaka and Watarai (1999)
PV in barotropic model

Breaking condition
\[ \frac{\partial q}{\partial y} < 0 \]
Saturation energy spectrum

\[ u < -\frac{\beta}{n^2 + l^2} = c \]

\[ |u| \approx |v| \quad \text{(Tanaka and Kasahara 1992)} \]

\[ E = \frac{1}{g} \int_0^{p_s} \frac{1}{2} (u^2 + v^2) dp \]

\[ = \frac{p_s}{g} c^2 = mc^2 \quad m = \frac{p_s}{g} \]

Mass for unit area

\[ \frac{\partial q}{\partial y} < 0 \quad \Rightarrow \quad E = mc^2 \]

Shepherd (1987)
Observed energy spectrum in $c$-domain

FGGE SOP1

Tanaka (1985)

$E = mc^2$

Tanaka and Kung (1988)
Global energy spectrum of $E = mc^2$

(Tanaka et al. 2004 GRL)

$$\frac{\partial q}{\partial y} < 0 \quad \Rightarrow \quad E = mc^2$$

$c$ Rossby phase speed
$m = \frac{p_s}{g}$ Mass of the air

Up-scale cascade
Rhines scale

Up-scale cascade

Total Energy Spectrum
JMA/GPV (DJF 2002/03)

Total Energy (Jm$^{-2}$)

Phase speed
Rhines scales on a sphere

\[ \frac{d w_i}{d \tau} = -i \sigma_i w_i - i \sum_{jk} r_{ijk} w_j w_k + f_i \]

\[ R_i = \frac{\sum |r_{ijk} w_j w_k|}{|\sigma_i w_i|} \]

**Rhines ratio**

**turbulence** \( R_i > 1 \)

**waves** \( R_i < 1 \)

**Rhines scale** \( R_i = 1 \)

(Tanaka et al. 2004 GRL)
Energy flux in $c$-domain

There is an energy source in the middle of the spectrum: inertial subrange theory fails.

Theory of Rossby wave saturation
Excitation of blocking and AO by up-scale energy cascade

(Tanaka and Terasaki 2004)
Low-frequency variability of the atmosphere
Summary

(1) Energy spectrum is examined in the 3D wavenumber domain using phase speed $c$.
(2) Energy spectrum of $E=mc^2$ is obtained and explained by Rossby wave saturation.
(3) Up-scale energy cascade to Rhine’s scale forms blocking.
(4) Further up-scale cascade to zonal energy forms the Arctic Oscillation.
Thank you.
Barotropic S-Model

- **3-D Spectral Model**

\[
\frac{dw_i}{d\tau} + i\sigma_i w_i = -i \sum_{jk} r_{ijk} w_j w_k + f_i, \\
i = 1, 2, 3, ..., \quad (7)
\]

where the symbols denote:
- \(w_i\): expansion coefficient of \(U\)
- \(f_i\): expansion coefficient of \(F\)
- \(\sigma_i\): Laplace's tidal frequency
- \(r_{ijk}\): nonlinear interaction coefficient
- \(\tau\): dimensionless time

- **Barotropic Spectral Model**

\[
\frac{dw_i}{d\tau} + i\sigma_i w_i = -i \sum_{jk} r_{ijk} w_j w_k + s_i, \\
i = 1, 2, 3, ..., \quad (m = 0), \quad (8)
\]

where the external forcing \(s_i\) includes barotropic-baroclinic interactions.
External Forcing

External forcing $s_i$ is statistically obtained by observed data using the least square method:

$$s_i = \bar{s}_i + A_{ij}w_j + B_{ij}w_j^* + \epsilon_i.$$  \hspace{2cm} (9)

where $\bar{s}_i$ is the climate of $s_i$ and the matrices $A_{ij}$ $B_{ij}$ are evaluated by minimizing $\epsilon_i$:

$$A_{ij} = s_i^{\dagger}w_j^+. \hspace{2cm} (10)$$

where $s_i' = s_i - \bar{s}_i$ and pseudo-inverse of $w_j$ is

$$w_j^+ = w_j^H \left( w_j w_j^H \right)^{-1}.$$  \hspace{2cm} (11)

The matrix $B_{ij}$ is similarly obtained stepwise by minimizing the first residual $\delta_i$:

$$B_{ij} = \delta_i w_j^{-+}. \hspace{2cm} (12)$$

Finally, the external forcing is given by

$$s_i = \bar{s}_i + A_{ij}w_j + B_{ij}w_j^* + (BC)_{ij}w_j + (DF)_{ij}w_j + (DZ)_{ij}w_j + (DE)_{ij}w_j.$$  \hspace{2cm} (13)

Physical processes considered in the S-Model:

(BC): baroclinic instability

(DF): biharmonic diffusion

(DZ): zonal surface stress and

(DE): Ekman pumping.