An HEVI time-splitting discontinuous Galerkin scheme for non-hydrostatic atmospheric modeling\textsuperscript{1}

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Outline

1 Motivation & Introduction
2 2D Euler System with orography
3 DG discretization
4 HEVI time-splitting scheme
5 Numerical Results
6 Summary
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Motivation

1. Peta-scale Super Computing Resources.
2. Atmospheric Model in Non-Hydrostatic Regime.
3. Requirements for discretization methods
   - Existing methods have serious limitations to satisfy all of the following properties:
     1. Local and global conservation
     2. High-order accuracy
     3. Computational efficiency
     4. Geometric flexibility ("Local" method, AMR)
     5. Non-oscillatory advection (monotonic, positivity preservation)
     6. High parallel efficiency (Petascale capability)
   - Discontinuous Galerkin Method (DGM) is a potential candidate
4. Efficient Time Integration Scheme Greatly Needed.
   - HEVI—horizontally explicit and vertically implicit is a good option.
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Based on conservation of momentum, mass and potential temperature (without Coriolis effect) the classical compressible 2D Euler system can be written in vector form:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) &= -\rho g \mathbf{k} \\
\frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta \mathbf{u}) &= 0
\end{align*}
\]
Idealized Non-Hydrostatic Atmospheric Model:

- Based on conservation of momentum, mass and potential temperature (without Coriolis effect) the classical compressible 2D Euler system can be written in vector form:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0 \\
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u + p I) &= -\rho g k \\
\frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta u) &= 0
\end{align*}
\]

- Removal of hydrostatic balanced state.

\[
\frac{d \bar{p}}{dz} = -\bar{\rho} g
\]
Terrain-Following $z$-Coordinates

Physical Grid $(x, z)$

Computational Grid $(x, \zeta)$

$\zeta = z_T$

$x, \zeta$ Computational Domain

$\zeta = 0$

$x$ (km)

$z_T - h$

$z_T - h$, $z(x) = h(x) + \zeta(z_T - h)$

$\zeta$, $h(x) \leq z \leq z_T$

$G = \frac{dz}{d\zeta}$

$G_{ij} = [0 \frac{d\zeta}{dx} 0 0]$;

$\tilde{w} = \frac{d\zeta}{dt} = \frac{1}{\sqrt{G}}(w + \sqrt{GG^{1/2}}u)$

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$^2$Gal-Chen & Somerville, JCP (1975)

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HEVI Time Splitting Scheme

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Terrain-Following $z$-Coordinates

- $(x, \zeta)$ coordinates.

\[
\zeta = z_T \frac{z - h}{z_T - h}, \quad z(\zeta) = h(x) + \zeta \frac{(z_T - h)}{z_T}; \quad h(x) \leq z \leq z_T
\]

- The metric terms (Jacobians) and new vertical velocity $\tilde{w}$ are

\[
\sqrt{G} = \frac{dz}{d\zeta}, \quad G^{ij} = \begin{bmatrix} 0 & \frac{d\zeta}{dx} \\ 0 & 0 \end{bmatrix}; \quad \tilde{w} = \frac{d\zeta}{dt} = \frac{1}{\sqrt{G}} (w + \sqrt{G} G^{12} u)
\]

---

2Gal-Chen & Somerville, JCP (1975)
In the transformed \((x, \zeta)\) coordinates, the Euler 2D system becomes\(^3\):

\[
\frac{\partial}{\partial t} \begin{bmatrix} \sqrt{G}\rho' \\ \sqrt{G}\rho u \\ \sqrt{G}\rho w \\ \sqrt{G}(\rho \theta)' \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \sqrt{G}\rho u \\ \sqrt{G}(\rho u^2 + p') \\ \sqrt{G}\rho w \\ \sqrt{G}\rho u \theta \end{bmatrix} + \frac{\partial}{\partial \zeta} \begin{bmatrix} \sqrt{G}\rho \dot{w} \\ \sqrt{G}(\rho \dot{w} + G^{12} p') \\ \sqrt{G}\rho \dot{w} w \\ \sqrt{G}\rho \dot{w} \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{G}\rho' g \end{bmatrix}.
\]

\(^3\)Skamarock & Klemp (2008), Giraldo & Restelli, JCP (2008)
\(^4\)Norman et al., JCP (2010)
\(^5\)Schär (2002), Klemp (2011)
In the transformed \((x, \zeta)\) coordinates, the Euler 2D system becomes\(^3\):

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\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} \sqrt{G} \rho' \\ \sqrt{G} \rho u \\ \sqrt{G} \rho w \\ \sqrt{G} (\rho \theta)' \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \sqrt{G} \rho u \\ \sqrt{G} (\rho u^2 + p') \\ \sqrt{G} \rho w \\ \sqrt{G} \rho u \theta \end{bmatrix} + \frac{\partial}{\partial \zeta} \begin{bmatrix} \sqrt{G} \rho \tilde{w} \\ \sqrt{G} (\rho \tilde{w} + G^{12} p') \\ \sqrt{G} \rho \tilde{w} \theta \\ \sqrt{G} \rho \tilde{w} \theta \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ -\sqrt{G} \rho' g \\ 0 \end{bmatrix}.
\end{align*}
\]

In Cartesian Coordinates (no orography)\((\sqrt{G} = 1, G^{12} = 1; \tilde{w} = w)\)\(^4\):

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} \rho' \\ \rho u \\ \rho w \\ (\rho \theta)' \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 + p' \\ \rho w \\ \rho u \theta \end{bmatrix} + \frac{\partial}{\partial z} \begin{bmatrix} \rho w \\ \rho w u \\ \rho w^2 + p' \\ \rho w \theta \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ -\rho' g \\ 0 \end{bmatrix}.
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In the transformed \((x, \zeta)\) coordinates, the Euler 2D system becomes\(^3\):

\[
\frac{\partial}{\partial t} \begin{bmatrix}
\sqrt{G} \rho' \\
\sqrt{G} \rho u \\
\sqrt{G} \rho w \\
\sqrt{G} (\rho \theta)' 
\end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix}
\sqrt{G} \rho u \\
\sqrt{G} (\rho u^2 + p') \\
\sqrt{G} \rho u w \\
\sqrt{G} \rho u \theta 
\end{bmatrix} + \frac{\partial}{\partial \zeta} \begin{bmatrix}
\sqrt{G} \rho \tilde{w} \\
\sqrt{G} (\rho \tilde{w} \tilde{w} + G^{12} p') \\
\sqrt{G} \rho \tilde{w} w \\
\sqrt{G} \rho \tilde{w} \theta 
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
-\sqrt{G} \rho' g 
\end{bmatrix}.
\]

In Cartesian Coordinates (no orography)\((\sqrt{G} = 1, G^{12} = 1; \tilde{w} = w)\)\(^4\):

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\rho u \\
\rho w \\
(\rho \theta)'
\end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix}
\rho u \\
\rho u^2 + p' \\
\rho u w \\
\rho u \theta 
\end{bmatrix} + \frac{\partial}{\partial z} \begin{bmatrix}
\rho w \\
\rho w u \\
\rho w^2 + p' \\
\rho w \theta 
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
-\rho' g 
\end{bmatrix}.
\]

Alternative formulations are also possible \(^5\) for \(\zeta\), but the system of equations remains in flux-form.

\[
\frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U)
\]

where \(U = [\sqrt{G} \rho', \sqrt{G} \rho u, \sqrt{G} \rho w, \sqrt{G} (\rho \theta)']^T\)

\(^3\)Skamarock & Klemp (2008), Giraldo & Restelli, JCP (2008)
\(^4\)Norman et al., JCP (2010)
\(^5\)Schär (2002), Klemp (2011)
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Consider a generic form of Euler’s System in two dimension.

\[ \frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U), \quad \text{in} \quad D \times (0,t_T); \forall (x,y) \in D \]

where \( U = U(x,y,t), \nabla \equiv (\partial/\partial x, \partial/\partial y), \ F = (F_1,F_2) \) is the flux function.
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where \( U = U(x, y, t) \), \( \nabla \equiv (\partial / \partial x, \partial / \partial y) \), \( F = (F_1, F_2) \) is the flux function.
Discontinuous Galerkin (DG) Components

Consider a generic form of Euler’s System in two dimension.

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\]

where \( U = U(x, y, t), \ \nabla \equiv (\partial / \partial x, \partial / \partial y), \ F = (F_1, F_2) \) is the flux function.

Weak Galerkin formulation:

\[
\frac{\partial}{\partial t} \int_{I_{i,j}} U_h \varphi_h \, ds - \int_{I_{i,j}} F(U_h) \cdot \nabla \varphi_h \, ds + \int_{\partial I_{i,j}} F(U_h) \cdot \bar{n} \varphi_h \, d\Gamma = \int_{I_{i,j}} S_h \varphi_h \, ds
\]
Consider a generic form of Euler’s System in two dimension.

\[
\frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U), \quad \text{in} \quad D \times (0, t_T); \quad \forall (x, y) \in D
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**Weak Galerkin formulation:**

\[
\frac{\partial}{\partial t} \int_{I_{i,j}} U_h \varphi_h \, ds - \int_{I_{i,j}} F(U_h) \cdot \nabla \varphi_h \, ds + \int_{\partial I_{i,j}} \hat{F}(U_h) \cdot \vec{n} \varphi_h \, d\Gamma = \int_{I_{i,j}} S_h \varphi_h \, ds
\]
The resulting form of DG-NH model is a system of ODEs.
\[
dU_h(t) = L(U_h), \quad t \in (0, t_T)
\]
The resulting form of DG-NH model is a system of ODEs.

\[
\frac{dU_h}{dt} = L(U^h), \quad t \in (0, t_T)
\]
Challenges for ODE system

Options & Challenges

- Explicit time integration efficient and easy to implement.
  Stringent CFL constraint $\Rightarrow$ tiny $\Delta t$, limited practical value.

\[
\frac{C\Delta t}{\bar{h}} < \frac{1}{2N + 1}
\]

- Strong Stability-Preserving (SSP)-RK.

<table>
<thead>
<tr>
<th>Heun's method</th>
<th>Explicit Runge-Kutta (SSP-RK3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2-stage 2\textsuperscript{nd} order)</td>
<td>(3-stage 3\textsuperscript{rd} order)</td>
</tr>
<tr>
<td>0 0 1 1 0</td>
<td>0 0 1 1 0</td>
</tr>
<tr>
<td>0 1 \frac{1}{2} \frac{1}{2} 1</td>
<td>0 1 \frac{1}{6} \frac{1}{6} \frac{2}{3}</td>
</tr>
</tbody>
</table>

HEVI: horizontally explicit and vertically implicit.
Challenges for ODE system

Options & Challenges

- Explicit time integration efficient and easy to implement. Stringent CFL constraint $\Rightarrow$ tiny $\Delta t$, limited practical value.

$$\frac{C\Delta t}{\bar{h}} < \frac{1}{2N + 1}$$

- Implicit time integration, unconditionally stable but generally expensive to solve. Overall efficiency still questionable.
Challenges for ODE system

Options & Challenges

- Explicit time integration efficient and easy to implement. Stringent CFL constraint $\Rightarrow$ tiny $\Delta t$, limited practical value.

  $$\frac{C\Delta t}{\bar{h}} < \frac{1}{2N + 1}$$

- Implicit time integration, unconditionally stable but generally expensive to solve. **Overall efficiency still questionable.**

- Semi-implicit time integration
  - Implicit solver for linear part and explicit solver for nonlinear parts. Needs **smart Helmholtz solver**.
  - **HEVI**: horizontally explicit and vertically implicit.
DG-NH Time Stepping-HEVI

For the resulting ODE system

\[ \frac{dU_h}{dt} = L(U^h), \quad \text{with} \quad C\Delta t < \frac{1}{2N+1} \]

To overcome \( \bar{h} = \min\{\Delta x, \Delta z\} \), treat the vertical time discretization (\( z \)-direction) in an implicit manner.

- **Benefit**: The effective Courant number is only limited by the minimum horizontal grid-spacing \( \min\{\Delta x, \Delta y\} \).
- **Bonus**: The ‘HEVI’ split approach might retain the parallel efficiency of HOMME for NH equations too.
- Horizontal part and vertical part connected by **Strang-type** time splitting, permitting \( \mathcal{O}(\Delta t^2) \) accuracy.
- **Remarks of HEVI**.
  - Particularly useful for 3D NH modeling (\( \Delta z : \Delta x = 1 : 1000 \)).
  - Global NH models adopt the HEVI philosophy, NICAM\textsuperscript{6}, MPAS\textsuperscript{7} etc.
  - Recent high-order FV-NH\textsuperscript{8} models based on operator-split method.

\textsuperscript{6}Satoh et al. 2008
\textsuperscript{7}Skamarock et al. 2012
\textsuperscript{8}Norman et al. (JCP, 2011), Ulrich et al. (MWR, 2012)
DG-NH Time Stepping-HEVI

- The Euler system for $U = (\sqrt{G \rho'}, \sqrt{G \rho u}, \sqrt{G \rho \tilde{w}}, \sqrt{G (\rho \theta)'} )^T$ is split into horizontal ($x$) and vertical ($\zeta$ or $z$) components:

  \[
  \text{(Euler sys)} \quad \frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial x} + \frac{\partial F_z(U)}{\partial z} = S(U)
  \]

  \[
  \text{(H-part)} \quad \frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial x} = S^x(U) = (0, 0, 0, 0)^T \tag{1}
  \]

  \[
  \text{(V-part)} \quad \frac{\partial U}{\partial t} + \frac{\partial F_z(U)}{\partial z} = S^z(U) = (0, 0, -\rho' g, 0)^T \tag{2}
  \]

- One possible option is to perform "$H-V-H$" sequence of operations:
  - Advance $H$-part by $\Delta t/2$ to get $U^*$, from the initial value $U^n$
  - Evolve $V$-part by a full time-step $\Delta t$, to obtain $U^{**}$ from $U^*$
  - Advance $H$-part with $U^{**}$ by $\Delta t/2$, to get the new solution $U^{n+1}$

- The vertical part may be solved implicitly with DIRK (Diagonally Implicit Runge-Kutta)\(^9\).

- For the implicit solver:
  - Inner linear solver uses Jacobian-Free GMRES (Most expensive part).
  - It usually takes 1 or 2 iterations for the outer Newton solver.

\(^9\)Durran, 2010
### General IMEX

For the semi-implicit RK method

We define $f^{\text{im}}(U(t), t) = L^V(U(t))$ and $f^{\text{ex}}(U(t), t) = L^H(U(t))$.

$$\frac{d}{dt} U_h = L^H(U_h) + L^V(U_h) \quad \text{in} \quad (t_n, t_{n+1}].$$

- Some popular choices of IMEX schemes,

<table>
<thead>
<tr>
<th>$c^{\text{ex}}$</th>
<th>$A^{\text{ex}}$</th>
<th>$b^T$</th>
<th>$c^{\text{im}}$</th>
<th>$A^{\text{im}}$</th>
<th>$b^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$1 - 2\alpha$</td>
<td>$\alpha$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$1 - \alpha$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{8}{7}$</td>
<td>$\frac{8}{7}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td></td>
<td></td>
<td>$\frac{120}{252}$</td>
<td>$\frac{71}{252}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\frac{3}{4}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{3}{4}$</td>
</tr>
</tbody>
</table>

- Semi-implicit Runge-Kutta (IMEX2)  
  2-stage 2\textsuperscript{nd} order, $\alpha = 1 - \frac{1}{\sqrt{2}}$

- Third order IMEX (IMEX3, SIRK-3A)  
  (3-stage 3\textsuperscript{rd} order, $\alpha = \frac{5589}{6524} + \frac{75}{233}, \beta = \frac{7691}{26096} - \frac{26335}{78288} + \frac{65}{168}$)
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Inertia Gravity Wave

Parameters

- Widely used for testing time-stepping methods in NH models
- Usually, $\Delta z \ll \Delta x$

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10 Skamarock & Klemp (1994)
Numerical Results

Inertia Gravity Wave

\[ \Delta t = 0.04 \text{ s for explicit RK-DG} \]
\[ \Delta t = 0.4 \text{ s for HEVI-DG} \]

\[ \Delta x = 500m, \Delta z = 50m \]

\[ P^2-GL \text{ grid.} \]

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10Skamarock & Klemp (1994)
Inertia Gravity Wave Convergence Study

The Courant number for HEVI-DG is only constrained by horizontal grid-spacing \((dx)\).

- \(\Delta x = 10\Delta z\)
- \(\Delta t\) for HEVI equals \(10\Delta t\) for RK2.
**Straka Density Current**

- $\Delta t = 0.075 \text{ s (both RK2 and HEVI)}$, Diffusion Coeff $\nu = 75.0 m^2/s$. Handled by LDG.
Numerical Results

**Straka Density Current**

- Grid convergence: No noticeable changes in the fields at 100 m or higher resolutions

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**Straka et al. (1993)**

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HEVI Time Splitting Scheme

April 8th, 2014
Potential Thermal Temperature Perturbation

- $\Delta z \approx 222$ m, $\Delta x \approx 832$ m, $\Delta t = 0.15$ s (HEVI)
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Conclusion & Future Work

1. Moderate-order \((P^N, N = \{2, 3, 4\})\) DG-NH model performs well for benchmark test cases.

2. HEVI time-splitting effectively relaxes the CFL constraint to the horizontal dynamics only, and permits larger time-step.

Future work.
- Incorporate HEVI in HOMME for full 3D DG-NH model
- Improve the efficiency for the horizontal part: multi-rate time integration scheme, subcycling.
- Adopt proper preconditioning process for efficient implicit solver in the vertical part.
- Test Hybrid DG for HEVI framework (Vertical Implicit Solver, Block Tri-diagonal Matrix, Reduce the degrees of freedom)
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1. Moderate-order $(P^N, N = \{2, 3, 4\})$ DG-NH model performs well for benchmark test cases.

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   - Incorporate HEVI in HOMME for full 3D DG-NH model
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Thank you!

Questions?

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