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The MOISE Inria team

Numerical Models
Local mesh refinement, ocean-atmosphere coupling, numerical methods

E. Blayo, L. Debreu, F. Lemarié

Variational Data assimilation
F.-X. LeDimet, A. Vidard, E. Kazantsev, M. Nodet

Uncertainty quantification
C. Prieur, C. Helbert

2. On the stability and accuracy of the harmonic and biharmonic isoneutral mixing operators in ocean models, Lemarié et al (Ocean Modelling, 2012)

3. Other problems and perspectives, Demange et al (in preparation)
Outline


2. On the stability and accuracy of the harmonic and biharmonic isoneutral mixing operators in ocean models, Lemarié et al (Ocean Modelling, 2012)

3. Other problems and perspectives, Demange et al (in preparation)
Spurious diapycnal mixing: The problem and a solution

- Tracer mixing in a stratified ocean (away from turbulent boundary layers)
  - $\kappa_{\text{dia}} \approx 10^{-5} \text{ m s}^{-1}$ (e.g. Ledwell et al., 1993)
  - $\kappa_{\text{iso}} \approx 10^3 \text{ m s}^{-1}$ (horizontal scale $\approx 100 \text{ km}$)

Vertical coordinates systems:

Main problems of terrain following ($\sigma$) (and geopotential $z$) models:
- Pressure gradient error and Diapycnal Mixing
- Stronger in ocean models than in atmospheric models
Spurious diapycnal mixing: The problem and a solution

ROMS model, terrain following coordinates, Third order upstream biased scheme for tracers. Salinity at 1000m depth

The (implicit) diffusion of the upstream biased scheme acts along horizontal coordinates . . .
Spurious diapycnal mixing: The problem and a solution

ROMS model, $\sigma$ coordinates, Third order upwind scheme for tracers
Salitiny at 1000m after 2 years of integration

Upstream order 3  Upstream order 5  Weno 5
A solution

Rotate the diffuse part of the upstream biased scheme:

\[
\frac{\partial q}{\partial x} \bigg|_{(2n+1)\text{order}} = \frac{\partial q}{\partial x} \bigg|_{(2n+2)\text{order}} + (-1)^n K \frac{\partial^{(2n+2)} q}{\partial x^{2n+2}} \text{rotated}
\]
Numerical issue

Rotation of the diffusive part introduces mixed (horizontal/vertical) and purely vertical derivatives

Stabilization of a (high order) rotated diffusion operator
Outline


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Isoneutral mixing problem (continuous formulation)

▷ Isoneutral Laplacian operator

Tracer field $q$ in an unbounded domain $\Omega = \mathbb{R}^3$

\[
\begin{aligned}
\partial_t q &= \mathcal{D}_2(q) = \nabla \cdot (\mathbf{R} \nabla q) = -\nabla \cdot \mathbf{F} \quad \text{in } \Omega \times [0, T], \\
q\big|_{t=0} &= q_0(x, y, z) \quad \text{in } \Omega,
\end{aligned}
\]

the matrix form of the tensor $\mathbf{R}$ is

\[
\mathbf{R} = \begin{pmatrix}
\kappa_x & 0 & \kappa_x \alpha_x \\
0 & \kappa_y & \kappa_y \alpha_y \\
\kappa_x \alpha_x & \kappa_y \alpha_y & \kappa_x \alpha_x^2 + \kappa_y \alpha_y^2
\end{pmatrix}, \quad \text{with} \quad \alpha = (\alpha_x, \alpha_y) = -\left(\frac{\partial_x \rho, \partial_y \rho}{\partial_z \rho}\right), \quad \|\alpha\| \ll 1.
\]

Properties

- Orthogonality condition: $\mathbf{F} \cdot \rho_\perp = 0$
- Satisfy monotonicity principle [Mathieu and Deleersnijder, 1998]
- Satisfy global tracer variance dissipation [Griffies et al., 1998]
Isoneutral mixing problem (continuous formulation)

▷ Isoneutral biharmonic operator

The rotated biharmonic operator reads

\[
\begin{align*}
\partial_t q &= \mathcal{D}_4(q) = -\mathcal{D}_2(\Psi) = -\nabla \cdot \mathbf{F}_4, \\
q\big|_{t=0} &= q_0(x, y, z),
\end{align*}
\]

with \(\Psi = \mathcal{D}_2(q)\),

composition of two isoneutral Laplacian operators

Properties

- Orthogonality condition: \(\mathbf{F}_4 \cdot \rho_\perp = 0\)
- Satisfy monotonicity principle \([\text{Mathieu and Deleersnijder, 1998}]\)
- Satisfy global tracer variance dissipation \([\text{Griffies, 2004}]\)
Time discretization

[Lemarié et al., 2012]

Objectives:
- same stability limit of the non-rotated operators
- only the vertical direction can be implicit

Proposed scheme:

→ Laplacian

\[ D_2(q) = \partial_x (\kappa_x [\partial_x q + \alpha_x \partial_z q]) + \partial_z (\alpha_x \kappa_x \partial_x q) + \partial_z (\kappa_x \alpha_x^2 \partial_z q) \]

\[
\begin{align*}
q^* & = q^n + \Delta t D_2(q^n) \\
q^{n+1} & = q^* + \theta \Delta t \left[ G_3(q^{n+1}) - G_3(q^n) \right]
\end{align*}
\]

Method of Stabilizing Corrections [van der Houwen & Verwer, 1979; Hundsdorfer, 2002]
Objectives:

- same stability limit of the non-rotated operators
- only the vertical direction can be implicit

Proposed scheme:

→ Biharmonic

\[
\begin{align*}
q^* &= q^n + \Delta t D_4(q^n) \\
q^{n+1} &= q^* + \Delta t \partial_z [\tilde{\kappa} \partial_z q^{n+1} - \tilde{\kappa} \partial_z q^n]
\end{align*}
\]

\(\tilde{\kappa}\) chosen through linear stability analysis

\[
\tilde{\kappa} = 8 \Delta z^2 \sigma_x s_x^2 (1 + s_x^2) / \Delta t \quad \Rightarrow \quad \sigma_x \leq \frac{1}{8}
\]

with 

\[
\begin{align*}
s_x &= \alpha_x \frac{\Delta z}{\Delta x}, \\
\sigma_x &= \kappa_x \frac{\Delta t}{\Delta x^4}
\end{align*}
\]
Conclusions of Lemarié et al (2012)

- The rotated operators can be advanced with the same time step as the non-rotated ones!!
- The rotated biharmonic can be a viable operator for use in high-resolution global models
- Clipping/tapering should act on $s$ rather than $\alpha$
- A slope-dependent discretization provides more accuracy notably for large grid slope ratios
- Has been implemented in ROMS and NEMO (also for rotation of hyperviscosity)

Alternatives to the use of rotated operators:

- No major improvements so far with higher-order advection schemes ...
- Arbitrary-Lagrangian-Eulerian (ALE) vertical coordinate $\rightarrow$ strong ongoing effort

Remaining open issue:

- Monotone advection schemes and diapycnal mixing

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Other sources of artificial diffusion (and dispersion)

- Amount of numerical viscosity and vertical advection schemes
- Barotropic/Baroclinic time splitting
- Internal Gravity Wave propagation
Other sources of artificial diffusion (and dispersion)

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Other sources of artificial diffusion (and dispersion)

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Shchepetkin and McWilliams (2005)

Internal Gravity Wave propagation

Baroclinic Jet

KE spectrum
Other sources of artificial diffusion (and dispersion)

- Amount of numerical viscosity and vertical advection schemes
- Barotropic/Baroclinic time splitting
- Internal Gravity Wave propagation

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial p}{\partial z} &= -\rho g \\
\frac{\partial \rho}{\partial t} + w \frac{\partial \bar{\rho}}{\partial z} &= 0
\end{align*}
\]

Very basic numerics:
- Second order discretization
- Computational mode of the Lorenz grid
- No dissipation

Decomposition into vertical modes and characteristic variables

\[
\begin{align*}
\rho(x, z, t) &= -\rho_0 \sum h_n(x, t) \frac{dM_n(z)}{dz} \\
\frac{\partial}{\partial t} (u_n \pm \frac{g}{c_n} h_n) + c_n \frac{\partial}{\partial x} (u_n \pm \frac{g}{c_n} h_n) &= 0
\end{align*}
\]
Other sources of artificial diffusion (and dispersion)

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Monotonicity of characteristic variables in the primitive variables formulation?

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} &= \frac{c_1 \Delta x}{2} \frac{\partial^2 u}{\partial x^2} \\
\frac{\partial u}{\partial w} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial p}{\partial z} &= -\rho g \\
\frac{\partial \rho}{\partial t} + w \frac{\partial \bar{\rho}}{\partial z} &= \frac{c_1 \Delta x}{2} \frac{\partial^2 \rho}{\partial x^2}
\end{align*}
\]
Internal Wave propagation


Demange J., Debreu L., Marchesiello P., 2014: On the use of a depth-dependent barotropic mode for free surface ocean models, to be submitted

Demange J., Debreu L., Marchesiello P., 2014: Vertical mode decomposition and dissipation, to be submitted
Properties of the discretized operator?

\[
\mathcal{D}_2(q) = \partial_x \left( \kappa_x \frac{\mathcal{J}_x(q, \rho)}{\partial_z \rho} \right) - \partial_z \left( \kappa_x \frac{\partial_x \rho}{\partial_z \rho} \frac{\mathcal{J}_x(q, \rho)}{\partial_z \rho} \right)
\]

\[
\mathcal{J}_x^u = (\partial_x q) \left( \partial_z \rho \right) - (\partial_x \rho) \left( \partial_z q \right), \quad \mathcal{J}_x^w = \left( \partial_x \rho \right) \left( \partial_x q \right) - \left( \partial_x q \right) \left( \partial_z q \right).
\]
Properties of the discretized operator?

\[ D_2(q) = \partial_x \left( \kappa_x \frac{J_x(q, \rho)}{\partial_z \rho} \right) - \partial_z \left( \kappa_x \frac{\partial_x \rho \ J_x(q, \rho)}{\partial_z \rho} \right) \]

\[ J_x^u = (\partial_x q)(\partial_z \rho) - (\partial_x \rho)(\partial_z q), \quad J_x^w = (\partial_x q)(\partial_z \rho) - (\partial_x \rho)(\partial_z q). \]

<table>
<thead>
<tr>
<th>ref.</th>
<th>min-max</th>
<th>TVD(^1)</th>
<th>(F(\rho) = 0)</th>
<th>misc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>COX</td>
<td>[Cox, 1987]</td>
<td>no</td>
<td>no</td>
<td>no</td>
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<tr>
<td>TRIADS</td>
<td>[Griffies et al., 1998]</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>SW-TRIADS</td>
<td>[Lemarié et al., 2012]</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

\(^1\)Tracer Variance Dissipation

\[ \rightarrow \text{all linear schemes produce over/under shootings [Beckers et al., 2000]} \]
Spatial discretization and stability limits

**TRIADS**

\[
\begin{pmatrix}
-s_x/2 & s_x^2 & s_x/2 \\
1 & -2(1+s_x^2) & 1 \\
s_x/2 & s_x^2 & -s_x/2
\end{pmatrix}
\]

**TRIADS**

<table>
<thead>
<tr>
<th></th>
<th>TRIADS</th>
<th>SW-TRIADS</th>
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<tr>
<td><strong>Laplacian</strong></td>
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</tr>
<tr>
<td>Rotated</td>
<td>(\sigma_x(1+s_x^2) \leq 1/2)</td>
<td>(\sigma_x \max(1,s_x^2) \leq 1/2)</td>
</tr>
<tr>
<td>Non-rotated</td>
<td>(\sigma_x \leq 1/2)</td>
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<tr>
<td><strong>Biharmonic</strong></td>
<td></td>
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</tr>
<tr>
<td>Rotated</td>
<td>(\left(\sigma_x^{(4)}(1+s_x^2)\right)^2 \leq 1/8)</td>
<td>(\left(\sigma_x^{(4)} \max(1,s_x^2)\right)^2 \leq 1/8)</td>
</tr>
<tr>
<td>Non-rotated</td>
<td>(\left(\sigma_x^{(4)}\right)^2 \leq 1/8)</td>
<td></td>
</tr>
</tbody>
</table>

SW-TRIADS (\(\alpha_x > 0\))

\[
\begin{pmatrix}
0 & s_x^2 - s_x & s_x \\
1 - s_x & -2(1+s_x^2) & 1 - s_x \\
s_x & s_x^2 - s_x & 0
\end{pmatrix}
\]

SW-TRIADS (\(\alpha_x < 0\))

\[
\begin{pmatrix}
-s_x & s_x^2 + s_x & 0 \\
1 + s_x & -2(1+s_x^2) & 1 + s_x \\
0 & s_x^2 + s_x & -s_x
\end{pmatrix}
\]

with \(\sigma_x = \kappa_x \frac{\Delta t}{\Delta x^2}\), \(s_x = \alpha_x \frac{\Delta x}{\Delta z}\), \(\sigma_x^{(4)} = \sqrt{B_x} \frac{\sqrt{\Delta t}}{\Delta x^2}\).
Adaptive Mesh Refinement for ocean models


→ everything works fine !!!


→ refinement criterion ?
→ How to define the bathymetry across the grid hierarchy ?
Before doing adaptive mesh refinement, are we sure than fixed mesh refinement is doing well?

- Debreu and Blayo, 2008: Two-way embedding algorithms: a review. Ocean Dynamics
  - High order update schemes / conservation and refluxing
  - Coupling at the fast modes (barotropic modes) level
  A library implementing Berger and Oliger algorithm and a source-to-source (monogrid to multigrid) translator

Current developments in ocean models:
Different vertical grids and potentially different vertical coordinate systems between coarse and fine grids