Non-hydrostatic sound-proof equations of motion
for gravity-dominated compressible flows

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I. Approximate equations of motion
   Hamilton's principle and asymptotics
   Soundproofing: anelastic vs hydrostatic

II. A « semi-hydrostatic » approximation
   Derivation and interpretation
   « Elliptic » problem for non-hydrostatic pressure
   Accuracy: normal-mode analysis

III. Conclusions
The atmosphere: a gravity-dominated, compressible flow

**Characteristic scales**

- **Velocity**: Sound $c \sim 340\text{m/s}$  
  Wind $U \sim 30\text{m/s}$
- **Time**: Buoyancy oscillations $N \sim g/c \sim 10^{-2}\text{s}^{-1}$  
  Coriolis $f \sim 10^{-4}\text{s}^{-1}$
- **Length**: Scale height $H = c^2/g = 10\text{km}$  
  Rossby radius $R = c/f \sim 1000\text{km}$

- **Mach number**: $M = U/c \ll 1$
- **Scale separation**: $f/N \sim H/R \ll 1$
- **Small numbers** => asymptotics => approximate equations
- **But not at the expense of conservation**: energy, momentum, potential vorticity
Can we make approximations while always maintaining conservation laws?

- Reversible mechanical systems obey Hamilton's least action principle.
- Conservation laws result from a symmetry of the action.
- Hamilton's principle asymptotics: approximating the action instead of the equations of motion systematically produces approximate systems with all conservation laws.

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**Diagram:**

- **Hamilton's least action principle (HP)**
  - Noether's theorems
  - HP for ideal fluid (Lagrangian description)
  - HP for ideal fluid (Eulerian description)
  - Relabelling symmetry => Kelvin's theorem & conservation of potential vorticity

**Timeline:**

- Maupertuis: 1744
- Euler, Lagrange: 1746
- 1834: Hamilton's HP
- 1900: Noether's Theorems
- 1960: Eckart
- 1960: Newcomb
- 1996: Padhye & Morrison
- 2002: Holm, Marsden, Ratiu

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**Additional Notes:**

- Semi-hydrostatic equations of motion
- T. Dubos, LMD/IPSL
Least action principle in curvilinear coordinates
(Tort & Dubos, accepted by J. Atmos. Sci.)

Spherical geoid
Shallow-atmosphere
Traditional

(Quasi-)Hydrostatic
Anelastic
Pseudo-incompressible
Boussinesq

Anelastic (Ogura & Phillips)

Pseudo-incompressible (Durran; Pauluis)

Compressible
Spherical-geoid

Non-traditional shallow-atmosphere (Tort & Dubos, 2014a)

Traditional shallow-atmosphere (Phillips, 1966)

Quasi-hydrostatic (White & Wood, 2012)
(White & Wood, 2012)
(Tort & Dubos, 2014b)

Spherical-geoid Quasi-hydrostatic (White & Wood, 1995)

NT shallow-atmosphere quasi-hydrostatic (Tort & Dubos, 2014a)

Primitive equations (Richardson, 1922 ?)
Filtering acoustic waves: hydrostatic vs anelastic or **Charybdis vs Scylla**

(waves over isothermal atmosphere: e.g. Davies et al., 2003; Arakawa & Konor, 2009)

Semi-hydrostatic equations of motion

T. Dubos, LMD/IPSL
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III. Conclusions
Neglecting vertical acceleration implies hydrostatic balance but the converse is not true.

Constraining (slaving) the density field
- breaks the pressure-density feedback loop
- key to suppressing acoustic waves

- « anelastic » slaving incorrect at large scales
- « hydrostatic » slaving is correct

Neglecting vertical acceleration implies hydrostatic balance but the converse is not true.

Arakawa & Konor (2009)
- impose density through hydrostatic balance but retain vertical acceleration
- pressure is the sum of hydrostatic pressure and a non-hydrostatic deviation
- non-hydrostatic pressure determined from a Poisson-like problem

Variational implementation of this physical idea?
• hydrostatic balance is a holonomic constraint (velocity not involved)
• Let us introduce a Lagrange multiplier to impose it!

\[ \mathcal{L} = \int \left[ \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - gz - e \left( \frac{1}{\rho}, s \right) \right] \rho \, dx \, dy \, dz \\
- \int \left( \lambda \rho g + p_{qs} \frac{\partial \lambda}{\partial z} \right) \, dx \, dy \, dz \quad p_{qs} = p(\rho, s) \]
- hydrostatic balance is a holonomic constraint (velocity not involved)
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\mathcal{L} = \int \left[ \frac{x^2 + y^2 + z^2}{2} - gz - e \left( \frac{1}{\rho}, s \right) \right] \rho dx dy dz \\
- \int \left( \lambda \rho g + p_{qs} \frac{\partial \lambda}{\partial z} \right) dx dy dz \quad p_{qs} = p(\rho, s)
\]

\[
Z = z + \lambda
\]

\[
\mathcal{L} \sim \int \left[ \frac{x^2 + y^2 + \dot{Z}^2}{2} - gZ \right] - e \left( \frac{1}{\rho} \frac{\partial Z}{\partial z}, s \right) \right] \rho dx dy dz
\]
Dubos & Voitus (submitted)

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\[ - \int \left( \lambda \rho g + p_{qs} \frac{\partial \lambda}{\partial z} \right) \, dx \, dy \, dz \]

\[ p_{qs} = p(\rho, s) \]

Interpretation
- The true height of air parcels is \( Z = z + \lambda \)
- \( z \) is their hydrostatic height
- \( \lambda \) is a vertical non-hydrostatic displacement

\[ \mathcal{L} \approx \int \left[ \frac{\dot{x}^2 + \dot{y}^2 + \dot{Z}^2}{2} - gZ \right] - e \left( \frac{1}{\rho} \frac{\partial Z}{\partial z}, s \right) \right] \rho \, dx \, dy \, dz \]

Approximation is accurate if \( d\lambda/dz \ll 1 \) and either
- \( DZ/Dt \ll Dx/Dt \) : hydrostatic scales
- or \( D\lambda/Dt \ll Dz/Dt \) : hydrostatic velocity is an accurate estimate of true vertical velocity
The true height of air parcels is $Z = z + \lambda$

- $z$ is their hydrostatic height
- $\lambda$ is a vertical non-hydrostatic displacement

- coordinates $(x,y,z)$ are slightly curvilinear
- $\rho$ is the pseudo-density associated to $(x,y,z)$
- the true density and pressure are $\rho + \rho'$ and $p + p'$:

\[
\rho' = -\rho \frac{\partial \lambda}{\partial z}, \quad p' = \rho' \frac{\partial p}{\partial \rho} = -\rho c^2 \frac{\partial \lambda}{\partial z}.
\]

**Adveective form**

- Usual terms
- Non-hydrostatic pressure
- Pseudo-forces due to $(x,y,z)$ being curvilinear

\[
\frac{Du}{Dt} + \theta \nabla \pi_q + ge_z = -\theta \nabla \pi' - g \nabla \lambda
\]

- Included in AK09
- Neglected by AK09
The true height of air parcels is $Z = z + \lambda$

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$$

**Momentum budget**

Usual terms

$$
\partial_t (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p_{qs} + \rho g e_z = \nabla \left( \rho c^2 \frac{\partial \lambda}{\partial z} \right) - \nabla \left( p_{qs} \frac{\partial \lambda}{\partial z} \right) + \frac{\partial}{\partial z} \left( p_{qs} \nabla \lambda \right)
$$

Non-hydrostatic pressure

Pseudo-forces due to $(x,y,z)$ being curvilinear
« Elliptic » problem for the non-hydrostatic displacement

- Assuming rigid boundaries, Dirichlet boundary conditions $\lambda=0$
- Assuming flat boundaries, Dirichlet boundary conditions $w=0$
- $w=\frac{Dz}{Dt}$ obeys the same Richardson's equation as with hydrostatic equations
  (Richardson, 1922; see also Dubos & Tort, submitted to MWR)

\[
\mathcal{A} \cdot w + \mathcal{B} \cdot \mathbf{u}_H = 0,
\]

\[
\mathcal{A} : w \rightarrow \partial_z \left( \rho c^2 \partial_z w \right)
\]

\[
\mathcal{B} : \mathbf{u}_H \rightarrow \partial_z \left( \rho c^2 \partial_x \mathbf{u}_H + \mathbf{u}_H \cdot \partial_x p_{qs} \right) + g \partial_x \cdot \left( \rho \mathbf{u}_H \right)
\]

\[
\mathcal{B}^* : \lambda \rightarrow \partial_x \left( \rho c^2 \partial_z \lambda \right) - \partial_z \lambda \partial_x p - \rho g \partial_x \lambda
\]

- Now observe that the momentum budget is of the form
  \[
  \partial_t w - \frac{1}{\rho} \mathcal{A} \cdot \lambda = \text{r.h.s.}
  \]

- One time-differentiation yields:
  \[
  \partial_t \mathbf{u}_H - \frac{1}{\rho} \mathcal{B}^* \cdot \lambda = \text{r.h.s.}
  \]

\[
\begin{pmatrix}
\mathcal{B} \frac{1}{\rho} \mathcal{B}^* \\
\mathcal{A} & -\rho
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\partial_t w
\end{pmatrix}
= \text{r.h.s}
\]

- Not Poisson-like: hydrostatic constraint involves vertical derivative of density
- Information propagates horizontally no farther than $\sim$ scale height
Relative frequency error for waves in an isothermal atmosphere at rest

The semi-hydrostatic approximation corrects much of the errors present in the hydrostatic approximation, except for vertically-long waves (several scale heights)
Relative frequency error for waves in an isothermal atmosphere at rest

Inaccurate: \( \frac{\Delta \lambda}{\Delta t} \ll \frac{\Delta z}{\Delta t} \) fails

Hydrostatic scales
Non-hydrostatic scales

Semi-hydrostatic

Hydrostatic scales
Non-hydrostatic scales
Inaccurate: \( \frac{\Delta z}{\Delta t} \ll \frac{\Delta x}{\Delta t} \) fails

Hydrostatic scales
Non-hydrostatic scales

Hydrostatic scales
Non-hydrostatic scales

Hydrostatic scales
Non-hydrostatic scales

Semi-hydrostatic equations of motion
Relative frequency error for waves in an isothermal atmosphere at rest

Semi-hydrostatic equations of motion

T. Dubos, LMD/IPSL
Relative frequency error for waves in an isothermal atmosphere at rest

- Acoustic frequency close to N
- No scale separation

Semi-hydrostatic equations of motion

T. Dubos, LMD/IPSL
Laws of (inviscid) atmospheric motion can be expressed concisely and safely from Hamilton's principle of least action.

Variational principles can be helpful for:

- Classifying existing approximation:
  
  *Tort & Dubos (J. Atmos. Sci., accepted)*

- Deriving new, consistent approximations:

  *Tort & Dubos (QJRMS, 2014), Dubos & Voitus (J. Atmos. Sci., submitted)*

The semi-hydrostatic system:

- Conserves energy, momentum, potential vorticity
- Possesses a well-defined self-adjoint problem yielding NH pressure
- Is accurate from hydrostatic to NH scales
- Except horizontally short, vertically long gravity waves

- Has been derived for an arbitrary equation of state
- Can easily be extended to include: moisture, deep-atmosphere, etc.
Laws of (inviscid) atmospheric motion can be expressed concisely and safely from Hamilton's principle of least action.

Approximate laws of motion provide insight into:
- Origin / nature of forces
- Actual, independant degrees of freedom
- 'Slaving' relationships between dependent and independant DOFs

For atmospherically-relevant flow regimes, including non-hydrostatic,

the independent degrees of freedom of atmospheric motion are precisely those of the hydrostatic primitive equations.