Quasi-uniform grids using a spherical helix

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Quasi-uniform grids on the sphere

Saff and Kuijlaars 1997

• Chemistry: Stable molecular structure (buckminsterfullerene)

• Physics: Location of identical point charges (J. J. Thomson’s problem)

• Computation: Quadrature on the sphere and computational complexity

• Botany: Distribution of pores on pollen (Tammes’s problem)

• Viral morphology, crystallography etc.
Proposed approaches

- Geodesic grids (Williamson 1968; Sadourny et al. 1968)
  NB. Spring dynamics used in NICAM
  (Tomita et al. 2001; Tomita and Satoh 2004)
- Cubed sphere (Sadourny 1972; McGregor 1996)
- Reduced (Kurihara 1965; Hortal and Simmons 1991)
- Yin-Yang (Kageyama and Sato 2004; Purser 2004)
- Fibonacci (Swinbank and Purser 2006)
- Conformally mapped polyhedra (Purser and Rančić 2011)
Dissection of an icosahedron

- Division of each side of 20 equilateral triangles into $n$

- $N = 10n^2 + 2$ points in total

- With bisection of edges $n = 2^l$, $N = 10(2^l)^2 + 2 = 12, 42, 162, 642, 2562, 10242, ...$

Sadourny, Arakawa and Mintz (1968)
Spherical helix

- $\lambda = m\theta \mod 2\pi$
  
  $m \equiv d\lambda/d\theta$ (slope)

- The length of a segment kept equal to the spacing between adjacent turns

- No limitations on the number of grids
Spherical helix for spherical SOM
(self-organizing maps)

Nishio, Altaf-Ul-Amin, Kurokawa and Kanaya (2006)

\[ \lambda = 2\sqrt{N\theta} \mod 2\pi \quad (1) \]

Compute the spiral length \( L \) numerically and arrange neurons at equal intervals.

With \( L \approx 2m \) for large \( m = 2\sqrt{N} \) the ratio between the adjacent turns and the segment length is

\[ \frac{2\pi N}{m2m} = \frac{N\pi}{m^2} = \frac{\pi}{4} \neq 1 \quad (2) \]
Generalized spiral points

Rakhmanov, Saff and Zhou (1994)

\[ \theta_k = \arccos(h_k), \quad h_k = -1 + \frac{2(k - 1)}{N - 1}, \quad 1 \leq k \leq N \quad (3) \]

\[ \lambda_k = \left( \lambda_{k-1} + \frac{c}{\sqrt{N(1 - h_k^2)}} \right) \mod 2\pi \quad (4) \]

\[ c = 3.6 < \left( \frac{8}{\sqrt{3}} \right)^{1/2} = 3.809 \quad (5) \]
The best packing on the sphere

Saff and Kuijlaars (1997)

- Hexagons except for 12 pentagons in the optimal arrangement.
- The area of the hexagon with the unit distance is $\sqrt{3}/2$.
- Ignoring the pentagonal cells, assume the sphere is covered by hexagonal Dirichlet cells

$$N \frac{\sqrt{3}}{2} \delta_N^2 = 4\pi$$

Thus the scaling factor is $\delta_N = \left(\frac{8\pi}{\sqrt{3}}\right)^{1/2} N^{-1/2}$
Rakhmanov et al. (1994)

\[ \delta = (\lambda_k - \lambda_{k-1}) \sqrt{1 - h_k} \]  
(7)

\[ = (\lambda_k - \lambda_{k-1}) \sin \theta_k = \frac{c}{\sqrt{N}} \]  
(8)

\[ m = \frac{2\pi}{\delta} = \sqrt{\frac{3}{8\pi}} \pi \sqrt{N} = \sqrt{\frac{3}{2}} \sqrt{\pi N} \]  
(9)

The ratio between the interval of the adjacent turns and the segment length is

\[ \frac{2\pi \ N}{m \ 2m} = \frac{N \pi}{m^2} = \frac{2}{3} \neq 1 \]  
(10)
Spherical Helix

Spherical spiral

Bauer (2000)

\[ \theta_k = \arccos(h_k), \quad h_k = 1 - \frac{2k - 1}{N}, \quad 1 \leq k \leq N \quad (11) \]

\[ \lambda = \sqrt{N\pi\theta} \mod 2\pi \quad (12) \]

With \( L \approx 2m \) for large \( m = \sqrt{N\pi} \) the ratio between the adjacent turns and the segment length is

\[ \frac{2\pi}{m} \frac{N}{2m} = \frac{N\pi}{m^2} = 1 \quad (13) \]
Analytically exact spiral

Koay (2011)

A line element

\[ ds = \sqrt{1 + m^2 \sin^2 \theta} \, d\theta, \quad m \equiv \frac{d\lambda}{d\theta} \]  

is integrated to yield

\[ L(\pi) = 2E(-m^2), \quad E(l) \equiv \int_0^{\pi/2} \sqrt{1 - l \sin^2 \theta} \, d\theta \]  

\( E(l) \) is the complete elliptic integral of the second kind and \( L(\pi) \approx 2m \) for large \( m \).
Let us define our previous work as follows:

Please note that our definition of the elliptic integral of the second kind, denoted by $\int\sqrt{1-\sin^2 \theta} \, d\theta$, turns out to be $2\pi$ for large $m$. Due to this simple relationship, the spacing between adjacent turns of the spiral curve to be nearly equal for each segment along the spiral curve as an element of the desired point set. To ensure that the spacing between adjacent turns of the spiral curve to be equal to the length of each segment is exactly $2\pi$, we have the value of $m$.

The second step is to divide the spiral curve into $n$ segments. Due to this simple relationship, the spacing between adjacent turns of the spiral curve to be nearly equal for each segment along the spiral curve as an element of the desired point set. To ensure that the spacing between adjacent turns of the spiral curve to be equal to the length of each segment is exactly $2\pi$, we have the value of $m$.

Note also that

$$\lim_{m \to \infty} m^{1/2} \approx 2\pi.$$

The new spiral point set of 88 points and its Voronoi tessellation.

Fig. 1. A surface element and a line element on the unit sphere $S^2$. Above, please note here that $\Theta = m/\Theta_1$ for all $j$. For completeness, the spiral points in Cartesian coordinates, $x = (\cos \theta, \sin \theta, \sin \theta)$, and with the initial solution of $j = m/\Theta_1$, at the end of the first spiral segment should satisfy the following equation:

$$x_1 = m - j.$$ 

Similarly, we can find the midpoint of each segment but we will have to solve for $j$ in

$$x_{1/2} = m - j.$$ 

Further examples are shown in Fig. 2. The new spiral point set of 88 points and its Voronoi tessellation.

Fig. 2. Bauer (2000) and Koay (2011).
Energy minimization on a sphere

The generalized energy for $N$ points $\omega_N = \{x_1, x_2, \ldots, x_N\}$ on the sphere

$$E(\alpha, \omega_N) = \begin{cases} 
\sum_{1 \leq i < j \leq N} \log \frac{1}{|x_i - x_j|} & \text{if } \alpha = 0 \\
\sum_{1 \leq i < j \leq N} |x_i - x_j|^{\alpha} & \text{if } \alpha \neq 0
\end{cases}$$

(16)
Various measures

- $\alpha = 1$: Maximization of distance $E(1, \omega_N)$

- $\alpha = 0$: Minimization of the logarithmic energy $E(0, \omega_N)$ (maximization of the product of distances).
  Logarithmic extreme points

- $\alpha = -1$: J. J. Thomson’s problem.
  Minimization of energy $E(-1, \omega_N)$.
  Fekete points

- $\alpha \to \infty$: The best packing on the sphere
  (Tamme’s problem, the hard sphere problem).
  Maximization of the smallest distance among $N$ points.
Theoretical approximation

Rakhmanov, Saff and Zhou (1994)

\[
f(-1, N) = \frac{N^2}{2} - 0.55230N^{3/2} + 0.0689N^{1/2} \quad (17)
\]

\[
f(0, N) = -\frac{1}{4} \log \left( \frac{4}{e} \right) N^2 - \frac{1}{4} N \log N - 0.026422N + 0.13822 \quad (18)
\]

\[
f(1, N) = \frac{2}{3}N^2 - 0.40096N^{1/2} - 0.188N^{-1/2} \quad (19)
\]
Comparison of homogeneity

Compare norms with $N = 12, 42, 162, 642, 2562, 10242$

- Sadourny et al. (1968)
- Tomita and Satoh (1994)
- Rachmanov et al. (1994)
- Bauer (2000)
Logarithmic energy

Sadourny et al. (1968)
Tomita and Satoh (1994)
Rachmanov et al. (1994)
Bauer (2000)
Spherical Helix

Energy

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Distance

![Graph showing distance vs. number of grids](image)

- Sadourny et al. (1968)
- Tomita and Satoh (1994)
- Rachmanov et al. (1994)
- Bauer (2000)
The number of points within a radius

The points within $r < \pi/6$ with $N = 400$
Variance $N = 642$

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Variance $N = 2562$

Sadourny et al. (1968)
Tomita and Satoh (1994)
Rachmanov et al. (1994)
Bauer (2000)

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**Variance** $N = 10242$

![Graph showing variance over distance with labels for authors and years: Sadourny et al. (1968), Tomita and Satoh (1994), Rachmanov et al. (1994), Bauer (2000).]
“Untidiness”

Nishio et al. (2006)

Sadourny et al. (1968)
Tomita and Satoh (1994)
Rachmanov et al. (1994)
Bauer (2000)
Design choices

• Voronoi tessellation

• Approximate ME: quadrature with spherical harmonics

• 1D structure: interpolation. Semi-Lagrangian advection

• Weaknesses: 2D decomposition, local subdivision, ...
Summary

- Quasi-uniform grids can be easily generated with a spherical helix.

- Spherical helix grids are more uniform than geodesic grids in various measures.

- The ratio between the adjacent turns and the segment length is unity in Bauer (2000) and Koay (2011) and not in Rakhmanov et al. (1994) and Nishio et al. (1997).

- The spiral length is approximated in Bauer (2000) and computed with an iterative scheme without approximation in Koay (2011).

- Design choices remain for the use in dynamical cores.