Development of smooth and homogeneity of icosahedral grid using spring dynamics method

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Introduction

• Icosahedral grid is based on icosahedron

• Used in many AGCMs
  – High performance with high resolution.
There are many ways to generate Icosahedral grid

- **Recursive type**
  - baumgardner-frederickson 1985
  - Heikes and Randall 1995a
  - Stune et al 1996
  - Xu et al 2006

- **Non-recursive type**
  - Williamson 1968
  - Sadourny et al. 1968
  - Steppeler et al. 2008

- **Spring dynamics method**
  - Tomita et al. 2001, 2002

We focus on this
Our purpose is to generate high performance icosahedral grid!

1. **Large minimum-grid interval**
   - Because of CFL condition
2. **Small maximum-grid interval**
   - To decrease error
3. **Smooth (no-discontinuity)**
   - For stable calculation
4. **Locally isotropic**
   - Each triangles are nearly regular
5. **Applicable to any resolution**
   - Even for high resolution
Spring dynamics method (SPR)  

• Adjacent grid points are connected
• Initial location of grid points are arbitrarily given.

\[
 M \frac{d \omega_0}{dt} = \sum_{i=1}^{n} k(d_i - d) e_i - \alpha \omega_0,
\]
\[
 \frac{d \mathbf{r}_0}{dt} = \omega_0,
\]

Natural spring length: we can freely determine

Following Tomita et al 2002, we introduce \( \beta \)

\( \beta \): natural spring length normalized by averaged grid interval

Repulsive when \( \beta > 1 \), attractive when \( \beta < 1 \),
zero natural spring length when \( \beta = 0 \)
Location of grid-points largely depend on natural spring length $\beta$

- $\beta = 0$
  - Small grid interval
  - Concentrates around pentagon

- $\beta = 0.4$
  - Less concentration

- $\beta = 1.0$

- $\beta = 1.2$
  - Not small

Tomita et al. 2002
Angular mean resolution distribution

\( \beta = 0 \)

Much contrast

\( \beta = 1.2 \)

Less contrast

\( \beta = 1.2 \) is better for simulation

Tomita et al 2002
Spring dynamics grids are unstable for high resolution and large $\beta$!

To avoid collapse, $\beta$ should be decreased when resolution is high
- Tomita et al 2002 used $\beta = 1.2$ but it works only when $dx \geq 28$km.
- NICAM uses $\beta = 1.15$ but it works only when $dx \geq 3.5$km.
- When $dx \leq 1.7$km, $\beta$ should be less than 1.15
However, homogeneity ($d_{\text{MIN}} / d_{\text{MAX}}$) is worse when $\beta$ is small.

To generate $dx=400m$ grid, $\beta$ should be 1.05, and $d_{\text{MIN}} / d_{\text{MAX}} = 0.739$ ..... $\rightarrow$ less homogeneous
Newly proposed method resolves the problem!

Generation Method

1. Generate Spring dynamic grid with $\beta = 0$
2. Applies transformation by smooth analytic function around pentagon
Spring dynamic grid with $\beta = 0$

Concentration around pentagon!
Distribution of grid interval fits with map factor of Lambert Conformal Conic Projection (LCCP) with map angle of 300°.

Reason is shown later.
Transformation by analytic function

$\phi$: Angular distance from pentagon

$\phi$ is transformed by

$$\phi_{\text{modif}} = r_{\text{modif}}^{-1}(r_{L}(\phi_{\beta=0}))$$

where

$$r_{\text{modif}}(\phi) \equiv Ar_{A}(\phi) + (1-A)r_{L}(\phi)$$

$$A \equiv 0.5 \left[1 - \tanh\left(\frac{\phi - \phi_{c2}}{\Delta}\right)\right]$$

$$r_{A}(\phi) \equiv (1 - \cos \phi_{c1})^f (\sin \phi_{c1})^{-(f+1)} \sin \phi.$$

$$r_{L} \equiv C_{L}(1 - \cos \phi)^f (\sin \phi)^{-f}.$$

• Then proposed grid is generated!
Homogeneity defined as: \( \frac{d_{\text{MIN}}}{d_{\text{MAX}}} \)

- SPR \( \beta=1.2 \) is the 2nd
- Recursive is the best
- Proposed grid is the 3rd
Which is better?

Recursive:
- Larger maximum
- Larger minimum

Proposed grid:
- Smaller maximum
- Smaller minimum
Weighted homogeneity \( (d_{\text{MIN}} / d_{\text{MAX}}^2) \)

- This indicates cost-efficiency of simulation because
  - Maximum error may be proportional to \( d_{\text{MAX}} \)
  - Therefore, required grid points for some limited error is proportional to \( d_{\text{MAX}}^2 \)
  - Time step needed by CFL constraint is proportional to \( d_{\text{MIN}} \)
  - In total, \( d_{\text{MIN}} / d_{\text{MAX}}^2 \) means cost-efficiency of calculation
Weighted homogeneity (cost-efficiency)

\[ \frac{d_{\text{MIN}}}{d_{\text{MAX}}^2} \]

Proposed grid is the best
Williamson’s test case 5

15 days later
Same viscosity is used

Proposed grid

More stable!

Spring dynamics grid $\beta=1.2$ (Tomita et al 2002)

Noise appeared
Summary

Compared with proposed grid, the other grids are

<table>
<thead>
<tr>
<th>Grid</th>
<th>Regularity of triangles</th>
<th>Smoothness</th>
<th>Homogeneity</th>
<th>Weighted homogeneity</th>
<th>Stability</th>
<th>Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring grid ($\beta=1.2$)</td>
<td>worse</td>
<td>equal</td>
<td>better</td>
<td>worse</td>
<td>worse</td>
<td>worse</td>
</tr>
<tr>
<td>Recursive grid</td>
<td>worse</td>
<td>worse</td>
<td>better</td>
<td>worse</td>
<td>worse</td>
<td>worse</td>
</tr>
<tr>
<td>Non-recursive grid</td>
<td>Not examined</td>
<td>worse</td>
<td>worse</td>
<td>worse</td>
<td>Not examined</td>
<td>Not examined</td>
</tr>
</tbody>
</table>

Proposed grid is better than the other grid in many properties.
Distribution of grid interval fits with map factor of Lambert Conformal Conic Projection (LCCP) with map angle of 300°

Reason is shown later
reason

• In the case $\beta=0$, when all triangles are regular, potential energy is minimum. (proven)

• Since potential energy tend to be minimum, triangles might become regular. (speculation)

Imagine Lambert map filled with regular triangle grids $\rightarrow$ Lambert grid

• It might be resemble to Lambert grid (speculation)

• So, resolution distribution of spring grid with $\beta=0$ is similar to map factor of Lambert map.
In the case $\beta=0$, when all triangles are regular, potential energy is minimum.

From Heron’s formula, total area of whole triangles is

$$S = \sum_{j=1}^{N_T} \sqrt{s_j(s_j - \tilde{d}_{1j})(s_j - \tilde{d}_{2j})(s_j - \tilde{d}_{3j})},$$

It is rewritten as

$$S = \sum_{h=1}^{N_S} \frac{\sqrt{3}}{6} \alpha'_h d_h^2, \quad \text{With} \quad 0 < \alpha_j \leq 1$$

It is rewritten as

$$S = \sum_{j=1}^{N_T} \frac{\sqrt{3}}{12} (\tilde{d}_{1j}^2 + \tilde{d}_{2j}^2 + \tilde{d}_{3j}^2) \alpha_j,$$

Where

$$\alpha'_h \equiv \alpha_i + \alpha_j, \quad 0 < \alpha'_h \leq 1$$

On the other hand, potential energy is

$$PE = \sum_{h=1}^{N_S} \frac{1}{2} k d_h^2.$$

If all triangle are regular, $\alpha'_h$ is unity (maximum) because potential energy is minimum.
Therefore, if potential energy approaches to minimum, we speculate that each triangles of spring dynamics grid with $\beta=0$ approaches to regular.

In reality, it is true! The right figure is $\alpha'_H$. It approaches to 1 when resolution increases.
• Lambert conformal conic projection map is filled with regular triangular mesh.
• Since it is conformal, corresponding sphere is also filled with regular triangular mesh which has singular points at the north pole.
• It can be presumed to be resemble to spring dynamics grids with $\beta=0$ which is also composed of regular triangles.
• If the presumption is true, grid interval of spring dynamics grid with $\beta=0$ is proportional to the map factor.

Map factor \[ r_L \equiv C_L (1 - \cos \phi)^f (\sin \phi)^{-f}. \]
Recursive method

- Joode's surface, a triangle, is subdivided into smaller triangles recursively.}

Stuhne et al. 1996

FIG. 2. The computational mesh structures for refinement levels $l = 0$ through $l = 6$ of the basic icosahedron.
Recursive grid

- Original recursive grid is very discontinuous.
- In Xu et al. 2006, it is smoothed by Laplacian.

Color indicates grid interval.

Original recursive

Smoothed by Laplacian

Pudikiewis 2011
Non-recursive method

- Core二十面体の大三角形（ひし形）を等間隔の三角形（ひし形）に分割し、球面に投影。
- 辺の個所に不連続は残る。
- 最大・最小格子間隔比は大きい。

Sadourny et al.1968
本来の動機

・京コンピュータを使用した、計算科学研究機構のグランドチャレンジプロジェクト（2013年）
  －水平400m解像度での全世界球気象シミュレーションデモラン。（格子点数54万点）

しかし、チームが用いている既存の格子作成方法では不具合が生じるので、対処したい！