MASS CONSERVATION PROPERTIES OF CG/DG METHODS
on non-conforming dynamically adaptive meshes

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Objectives

Compare mass conservation properties of CG/DG methods using:

- static and dynamic non-conforming mesh refinement
- density current and rising thermal bubble test cases
- different polynomial order
- different resolution

Can CG perform as well as DG?
Unified CG DG method

\[ \frac{\partial q}{\partial t} + \nabla \cdot F(q) = S(q) \]

\[ \int_{\Omega_e} \psi_i \frac{\partial q}{\partial t} \, d\Omega_e + \int_{\Gamma_e} n \cdot (\psi_i F(q)) \, d\Gamma_e - \int_{\Omega_e} \nabla\psi_i \cdot F(q) \, d\Omega_e = \int_{\Omega_e} \psi_i S(q) \, d\Omega_e \]
Unified CG DG method

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Direct Stiffness Summation
Data structures - storage

CG

DG

global numbering

\[ Q \]

local numbering

\[ Q^T \]
DSS on non-conforming elements

CG storage

DG storage

\[ Q \]

\[ Q^T \]
DG flux on non-conforming elements

\[ \int_{-1}^{1} \left( q^{Lk}(z^{(k)}) - q^{L}(s \cdot z^{(k)} + o^{(k)}) \right) \psi_{i}(z^{(k)}) \, dz^{(k)} = 0 \]

\[ \int_{-1}^{1} (q^{R}(\xi) - \bar{q}^{R}(\xi)) \psi_{i}(\xi) \, d\xi = 0 \]

Test cases

Density current

Rising thermal bubble
Initial tests - no AMR DG

\[
m(t) = \sum_{e} \int_{\Omega_e} \rho(t) \, d\Omega_e
\]

\[
M_1 = \frac{|m(t_1) - m(0)|}{m(0)}
\]
Initial tests - no AMR DG

RK35 coefficients are slightly off !!!


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Initial tests - AMR DG

(a) uniform

(b) AMR $\theta_t = 0.001$

(c) AMR $\theta_t = 4.0$

(d) element count
Summing algorithm

Serial

Pairwise sum

\[ \text{Less roundoff error!} \]

Results - density current

- **Uniform Density Current**
  - Various density current distributions are shown with different markers and colors.
  - The x-axis represents time, ranging from 0 to 10,000.
  - The y-axis shows the density current, with values ranging from $10^{-13}$ to $10^{-15}$.

- **Static Density Current**
  - Similar to the uniform case, but with static conditions.
  - The graph shows stability and consistency in the density current over time.
  - The y-axis indicates density current values from $10^{-15}$ to $10^{-17}$.
Results - density current
Results - rising thermal bubble

![Graphs showing results for uniform and AMR resolutions with different methods and resolutions.](image-url)
Conclusions

• AMR affects mass conservation only slightly

• CG can conserve as well as DG in AMR simulations

• CG is less robust for AMR simulations

• devil is in the details!

To do

• Investigate the filter effect