Controlling the energy spectrum

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with Jason Frank & Ben Leimkuhler
Fig. 3. Variance power spectra of wind and potential temperature near the tropopause from GASP aircraft data. The spectra for meridional wind and temperature are shifted one and two decades to the right, respectively; lines with slopes $-3$ and $-5/3$ are entered at the same relative coordinates for each variable for comparison.
Incompressible Navier-Stokes

In a 2D periodic box:

\[ \omega_t + J(\psi, \omega) = f + \nu \Delta \omega - \alpha \omega, \]

where the vorticity \( \omega = \Delta \psi \) and

\[ J(\psi, \omega) = \psi_x \omega_y - \psi_y \omega_x. \]

The friction \(-\alpha \omega\) is restricted to the largest scales
Pseudo-spectral method

\[
\omega_k(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \omega(x, t) e^{-i k \cdot x} dx
\]

with the dynamics

\[
\dot{\omega}_k + J_k(\omega) = f_k + \nu \Delta_k \omega_k - \alpha \omega_k 1_{|k| \leq 3},
\]
Energy spectrum

\[ E_k = \frac{-1}{2} \sum_{k-1/2 < |k| < k+1/2} \Delta_k^{-1} \omega_k \omega_k^* \] (1)
Two-dimensional turbulent energy spectrum

mean energy spectra

\[ \langle E_k \rangle \sim \nu^{-1/2} \]

Friction forcing

\( \nu^{-1/2} \sim \text{viscous scales} \)

Full model with resolution \( 768^2 \)
Truncated energy spectrum

mean energy spectra

$\langle E_k \rangle$

Full model with resolution $768^2$

Full model with resolution $256^2$
Artificial viscosity

Increase $\nu$ such that the viscosity acts at a resolved scale.
Dispersivity

Compare spread of ensemble members to RMS error of ensemble members.

\[ s(t) = \langle |a_i - \bar{a}|^2 \rangle_i, \quad e(t) = \langle |a_i - A|^2 \rangle_i \]

\(a_i\) is ensemble observable, \(\bar{a}\) the ensemble mean, \(A\) the “truth”.
Auto-correlation functions

\[ R_{\omega \omega}(\tau) = \frac{1}{T} \int_{0}^{T} \omega(t + \tau)\omega(t) \, dt \]
Auto-correlation functions

$1 - R_{\omega \omega}(\tau)$
Truncation drawbacks

- Energy spectrum does not match the observed data
- Insufficient energy at small scales causes
  - underdispersive ensemble
  - overly time-correlated solutions
- Well-studied problem
  - Backscatter algorithms (Shutts, Berner et al.)
  - Stochastic subgrid models (Berloff, Marstorp et al, Crommelin & Vanden-Eijnden)
  - Cut-off filters (Tullock & Smith 2009)
  - Hyperviscosity (textbook)
Could *impose* the spectrum with constraints
Could *impose* the spectrum with constraints

\[
\dot{\omega}_k + J_k(\omega) = f_k + \nu \Delta_k \omega_k - \alpha \omega_k 1_{|k| \leq 3} - \sum_l \xi_l \partial \omega_k c_l(\omega)
\]

\[
0 = c_l(\omega) - C_l.
\]

A Differential Algebraic Equation with Lagrange multipliers \( \xi_l \)

\[c_l(\omega) = E_l(\omega), \quad C_l = \langle E_l \rangle_{\text{data}}\]

Too strict!
Correction device

- Only the average has to be controlled
- Similar to a thermostat in Molecular Dynamics
Correction device

- Only the average has to be controlled
- Similar to a thermostat in Molecular Dynamics
  - Nosé-Hoover thermostat

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -\nabla V(q) - \xi p \\
\mu \dot{\xi} &= K(p) - nk_B T
\end{align*}
\]
Correction device

- Only the average has to be controlled
- Similar to a *thermostat* in Molecular Dynamics
  - Nosé-Hoover thermostat

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\begin{align*}
\dot{q} &= p \\
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\mu \dot{\xi} &= K(p) - nk_B T
\end{align*}
\]

- Leimkuhler et al. ('08)

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -\nabla V(q) - \xi p \\
\mu \dot{\xi} &= \frac{1}{\int_0^t \phi(s) \, ds} \int_0^t \phi(t - s) K(p(s)) \, ds - nk_B T
\end{align*}
\]
Correction device

Combine previous ideas

\[ \dot{\omega}_k + J_k(\omega) = f_k + \nu \Delta_k \omega_k - \alpha \omega_k 1_{|k| \leq 3} - \sum_l \xi_l \partial \omega_k c_l(\omega) \]

\[ \mu \dot{\xi}_l = \frac{1}{\int_0^t \phi(s) \, ds} \int_0^t \phi(t - s) c_l(\omega(s)) \, ds - C_l. \]
Vorticity field snapshot

After \( t=1 \)
Vorticity field snapshot

After $t=10$

scaled viscosity

reference

with device
Spectrum

mean energy spectra

\[ \langle E_k \rangle \]

- Full model with resolution 768\(^2\)
- Full model with resolution 256\(^2\)
- Scaled viscosity model with resolution 256\(^2\)
- With correction device at resolution 256\(^2\)
Auto-correlation functions

\[ R_{\omega\omega}(\tau) = \frac{1}{T} \int_0^T \omega(t + \tau)\omega(t) \, dt \]
Auto-correlation functions

$1 - R_{\omega\omega}(\tau)$
Summary

- Truncation of length scales disturbs spectrum at the smallest resolved scales
- A correction is made that restores the energy spectrum
- This correction also improves dispersivity and decorrelation times