An energy-conserving quasi-hydrostatic deep-atmosphere dynamical core

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PDEs on the Sphere, NCAR, Co

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Motivations: LMD-Z and Planets

Atmospheric component of the IPSL model

Planetary atmosphere version e.g. Mars

Common dynamical core on a longitude/latitude grid:
- *shallow-atmosphere* hydrostatic equations (HPE),
- enstrophy-conserving scheme (*Sadourny, 1975a*).
Motivations : LMD-Z and Planets

Atmospheric component of the IPSL model

Planetary atmosphere version
e.g. Giant gas planets, Titan
→ DEEP ATMOSPHERES

The dynamical core requires a *deep-atmosphere* version.

**GOALS**

- solve the *deep-atmosphere* quasi-hydrostatic equations (QHE) (*White and Bromley, 1995*)
  AND the recently derived non-traditional *shallow-atmosphere* equations (NTE) with complete Coriolis force (*Tort and Dubos, 2014a*),
- preserve some discrete conservation properties.
An energy-conserving scheme (Lagrangian vertical coordinate case)
  Curl-form and Hamiltonian formulation
  How to conserve energy?
  Iterative procedure to solve hydrostasy

Application to LMD-Z
  Mass coordinate: how is that different from a Lagrangian vertical coordinate?
  Idealized test cases

Summary
An energy-conserving scheme (Lagrangian vertical coordinate case)
  Curl-form and Hamiltonian formulation
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Summary
Curl-form and Hamiltonian formulation

Derivation in curl-form using a time-dependent curvilinear coordinates system

- from White and Bromley (1995)’s equations (spherical coordinates and advective form),
- using a general vertical coordinate \( \eta : r(\lambda, \phi, \eta; t) \).

Momentum prognostic variables \( \tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}) \) (Dubos and Tort, 2014)

- shallow-atmosphere \( \tilde{\mathbf{u}} = (r_0 \cos \phi(u + \Omega r_0 \cos \phi), r_0 v) \),
- deep-atmosphere \( \tilde{\mathbf{u}} = (r \cos \phi(u + \Omega r \cos \phi), rv) \),

Momentum

\[
\partial_t \tilde{\mathbf{u}} + \frac{1}{\tilde{\rho}} (\nabla \times \tilde{\mathbf{u}}) \times \mathbf{U} + \nabla (K + \Phi) + \theta \nabla \pi = 0
\]

Mass

\[
\tilde{\rho} = \rho r^2 \cos \phi \partial_\eta r, \quad \partial_t \tilde{\rho} + \nabla \cdot \mathbf{U} = 0
\]

Entropy

\[
\Theta = \tilde{\rho} \theta, \quad \partial_t \Theta + \nabla \cdot (\theta \mathbf{U}) = 0
\]
Curl-form and Hamiltonian formulation

Variational interpretation

- Hamiltonian \( \mathcal{H}(\bar{\rho}, \bar{u}, \Theta, r) = \int_V d\lambda d\phi d\eta \bar{\rho} \left( K(\bar{u}, r) + \Phi(r) + e \left( \frac{\cos \phi \partial_\eta \bar{r}^3}{3\bar{\rho}}, \frac{\Theta}{\bar{\rho}} \right) \right) \)
- equations are written in term of functional derivatives of \( \mathcal{H} \)

Functional derivatives

\[
\begin{align*}
\frac{\delta \mathcal{H}}{\delta \bar{\rho}} &= K + \Phi \\
\frac{\delta \mathcal{H}}{\delta \bar{u}} &= U \\
\frac{\delta \mathcal{H}}{\delta \Theta} &= \pi \\
\frac{\delta \mathcal{H}}{\delta r} &= -\bar{\rho} \left( \frac{u^2 + v^2}{r} + 2\Omega \cos \phi u \right) + r^2 \cos \phi \partial_\eta p + \bar{p}g(r) \\
\end{align*}
\]

HYDROSTATIC CONSTRAINT : \( \frac{\delta \mathcal{H}}{\delta r} = 0 \)
How to conserve energy?

*Imitate the exact Hamiltonian formulation at discrete level*

*Salmon, 2004*

1. Choose the spatial grid

- horizontal : C-grid,
- vertical : Lorenz grid.
How to conserve energy?

2. Express the discrete energy budget, take care that the terms compensate.

\[
\frac{dH}{dt} = \sum \left[ \frac{\delta H}{\delta \rho} \partial_t \rho + \frac{\delta H}{\delta \Theta} \partial_t \Theta + \frac{\delta H}{\delta \tilde{u}} \partial_t \tilde{u} + \frac{\delta H}{\delta \tilde{v}} \partial_t \tilde{v} + \frac{\delta H}{\delta r} \partial_t r \right],
\]

\[
= -\sum \left[ \frac{\delta H}{\delta \rho} \left( \frac{\delta_i \delta H}{\delta \tilde{u}} + \frac{\delta_j \delta H}{\delta \tilde{v}} \right) + \frac{\delta H}{\delta \Theta} \left( \frac{\delta_i}{\theta_i} \frac{\delta H}{\delta \tilde{u}} + \frac{\delta_j}{\theta_j} \frac{\delta H}{\delta \tilde{v}} \right) + \frac{\delta H}{\delta \tilde{u}} \left( -\frac{\delta_i \tilde{v} - \delta_i \tilde{u}}{\tilde{\rho}^j} \frac{\delta H}{\delta \tilde{v}} + \delta_i \frac{\delta H}{\delta \rho} + \frac{\theta_i}{\theta_j} \frac{\delta H}{\delta \Theta} \right) \right]
\]

\[
+ \frac{\delta H}{\delta \tilde{v}} \left( \frac{\delta_i \tilde{v} - \delta_j \tilde{u}}{\tilde{\rho}^j} \frac{\delta H}{\delta \tilde{u}} + \delta_j \frac{\delta H}{\delta \rho} + \frac{\theta_i}{\theta_j} \frac{\delta H}{\delta \Theta} \right) - \frac{\delta H}{\delta r} \partial_t r \right] = 0.
\]

3. Discretize the Hamiltonian and deduce the discrete derivatives

\[
\mathcal{H} = \sum \tilde{\rho} \left( \frac{1}{2} \left( \frac{\tilde{u}}{\tilde{r}^{ik}} - \Omega \tilde{r}^{ik} \cos \phi \right)^2 \right) + \frac{1}{2} \left( \frac{\tilde{v}}{\tilde{r}^{jk}} \right)^2 + e \left( \frac{\cos \phi \delta_k r^3}{3 \tilde{\rho}}, \frac{\Theta}{\tilde{\rho}} \right) + \Phi(\tilde{r}^k)
\]
Iterative procedure to solve hydrostasy

<table>
<thead>
<tr>
<th>HPE</th>
<th>QHE</th>
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<tbody>
<tr>
<td>$\delta_k p = -\tilde{\rho}^k g$</td>
<td>$\delta_k p = -\tilde{\rho}g(r^k)^k$</td>
</tr>
<tr>
<td></td>
<td></td>
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</tbody>
</table>
| $a^2 \delta_k r = \frac{\tilde{\rho}_k \theta c_p p^k - 1}{p_F^k \cos \phi}$ | $\delta_k r^3 = \frac{3\tilde{\rho}_k \theta c_p p^k - 1}{p_F^k \cos \phi}$ |}

$r$ is the solution of a non-linear elliptic problem whose $p$ is a byproduct.

- HPE: direct obvious solution,
- QHE: iterative solution (fix point or Newton’s method).
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Summary
Mass coordinate: how is that different from a Lagrangian vertical coordinate?

Mass budget: \( \partial_t \tilde{\rho} + \partial_\lambda U + \partial_\phi V + \partial_\eta (\tilde{\rho} \dot{\eta}) \)

- \( \tilde{\rho} \) is not a prognostic variable anymore and \( \mathcal{H} \) is expressed with respect to integrated mass \( M_s \) instead of local mass \( \tilde{\rho} \),
- \( \dot{\eta} \neq 0 \), there is additional non-zero vertical transport in the equations.

Vertical relabeling symmetry (Dubos and Tort, 2014)

- compensation of vertical transports terms due to vertical relabeling symmetry,
- we may imitate to the discrete level this relabeling to cancel vertical transport in the discrete energy budget (Tort et al, in preparation).
Idealized test cases

Like-Earth planet experiment

Baroclinic instability  
*Ullrich et al, 2013*

Newtonian relaxation  
*Held and Suarez, 1994*

**FIG 4.** Surface pressure $p_s$ at day 10

**FIG 5.** Zonally averaged zonal velocity over 1000 planetary rotations
Idealized test cases

Small like-Earth planet experiment

**Non-traditional regime**

\[ X \sim U/(\Omega_e H) \Rightarrow X = 15 \]

**Deep-atmosphere regime**

\[ X \gg U/(\Omega_e H) \text{ e.g. } X = 50 \]

![Non-traditional formulation](image1.png)

![Deep-atmosphere formulation](image2.png)

**FIG 6. Zonally averaged zonal velocity over 1000 planetary rotations**

**FIG 7. Zonally averaged zonal velocity over 1000 planetary rotations**

development of a zonally averaged easterly flow in the tropics scaled by \( U = -2X\Omega_0 H \cos \phi \) ([White and Bromley, 1995, Wedi and Smolarkiewicz, 2009])
Summary

Implementation of the QHE into LMD-Z

- energy-conserving,
- systematic method to discretize hydrostatic systems,
- ongoing: newtonian relaxation on a Titan-like planet.


Consequence of vertical relabeling

Energy budget induced by vertical flux (mass-based coordinate) $W = \tilde{\rho} \tilde{\eta}$

$$\frac{dH}{dt} = \sum \left[ \frac{\delta H}{\delta \tilde{\rho}} \partial_t \tilde{\rho} + \frac{\delta H}{\delta \Theta} \partial_t \Theta + \frac{\delta H}{\delta \tilde{\eta}} \partial_t \tilde{\eta} + \frac{\delta H}{\delta \tilde{\nu}} \partial_t \tilde{\nu} + \frac{\delta H}{\delta r} \partial_t r \right] ,$$

$$= -\sum \left[ \frac{\delta H}{\delta \tilde{\rho}} \left( \delta_i \frac{\delta H}{\delta \tilde{\eta}} + \delta_j \frac{\delta H}{\delta \tilde{\nu}} + \delta_k W \right) + \frac{\delta H}{\delta \Theta} \left( \delta_i \left( \frac{\bar{\theta}^i \delta H}{\delta \tilde{\eta}} \right) + \delta_j \left( \frac{\bar{\theta}^j \delta H}{\delta \tilde{\nu}} \right) + \delta_k \left( \bar{\theta}^k W \right) \right) \right]$$

$$+ \frac{\delta H}{\delta \tilde{\eta}} \left( \frac{W^i \delta_k \tilde{u}}{\tilde{\rho}^i} - \frac{\delta_i \tilde{v} - \delta_i \tilde{u} \frac{\delta H}{\delta \tilde{\nu}}}{\tilde{\rho}^j} + \delta_i \frac{\delta H}{\delta \tilde{\rho}} + \bar{\theta}^i \delta_i \frac{\delta H}{\delta \Theta} \right)$$

$$+ \frac{\delta H}{\delta \tilde{\nu}} \left( \frac{W^j \delta_k \tilde{v}}{\tilde{\rho}^j} + \frac{\delta_i \tilde{v} - \delta_i \tilde{u} \frac{\delta H}{\delta \tilde{\nu}}}{\tilde{\rho}^i} + \delta_j \frac{\delta H}{\delta \tilde{\rho}} + \bar{\theta}^j \delta_j \frac{\delta H}{\delta \Theta} \right) - \frac{\delta H}{\delta r} \partial_t r \right] = 0 ?$$

How to cancel the blue terms? Bernoulli function $B = \frac{\delta H}{\delta \tilde{\rho}}$ is such as :

$$\delta_k B + \bar{\theta}^k \delta_k \frac{\delta H}{\delta \Theta} - \frac{1}{\tilde{\rho}^i} \frac{\delta H^i}{\delta \tilde{\eta}} \delta_k \tilde{u} - \frac{1}{\tilde{\rho}^j} \frac{\delta H^j}{\delta \tilde{\nu}} \delta_k \tilde{v} = 0$$

$$\sum_k \delta_k \beta B = \sum_k \delta_k \beta (K + \Phi)$$
Consequence of vertical relabeling

Tort and Dubos, 2014a

\[
\frac{Du}{Dt} - \left(2\Omega \left(1 + \frac{2z}{r_0}\right) + \frac{u}{r_0 \cos \phi}\right) v \sin \phi + 2\Omega \cos \phi w + \frac{1}{\rho r_0 \cos \phi} \frac{\partial p}{\partial \lambda} = 0
\]

\[
\frac{Dv}{Dt} + \left(2\Omega \left(1 + \frac{2z}{r_0}\right) + \frac{u}{r_0 \cos \phi}\right) u \sin \phi + \frac{1}{\rho r_0} \frac{\partial p}{\partial \phi} = 0
\]

\[
\delta_{NH} \frac{Dw}{Dt} + 2\Omega \cos \phi u + g + \frac{1}{\rho} \frac{\partial p}{\partial z} = 0
\]

Non-traditional shallow-atmosphere angular momentum is now conserved

\[a \cos \phi (u + \Omega r_0 \cos \phi (r_0 + 2z))\]
Stability analysis of the vertical discretization

Isothermal atmosphere at rest on the $f - F$-plane - free pressure surface BC
extension of Thuburn et al, 2002b’s work with a rigid lid BC.

**FIG 2.** Analytical vs numerical frequency spectrum

**FIG 3.** Numerical eigenvalues with a Lagrangian vs mass-based coordinate

- Lagrangian coordinate $1/\sigma \to \infty$,
- Mass-based vertical coordinate $1/\sigma \to 15$ years.