A Stability Analysis of Finite-Volume Advection Schemes Permitting Long Time Steps

Peter Hjort Lauritzen (pel@ucar.edu)

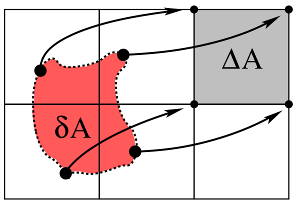
National Center for Atmospheric Research (NCAR), Boulder, Colorado, U.S.A.

NCAR

Flux-form and cell-integrated finite-volume schemes

Finite-volume advection schemes can be divided into two categories:

1. Semi-Lagrangian type schemes, referred to as **cell-integrated semi-Lagrangian (CISL)** schemes, in which the mass in cells moving with the flow is tracked (e.g., Nair et al. 2002; Zerroukat et al. 2002). In upstream CISL schemes cells that after one time step, Δt, end up at the regular (Eulerian) grid are considered, that is, mass over the shaded area in the Figure, referred to as the *departure cell* and with area δA, is tracked as it moves with the flow and ends up at the regular (Eulerian) cell, referred to as *arrival cell* and with area ΔA, located in upper right corner of the Figure.



The discretized CISL mass continuity equation is given by

$$\overline{\rho}(t=\Delta t) = \frac{1}{\Delta A}\int_{\delta A}\rho(t=0)\,dA, \qquad (1)$$

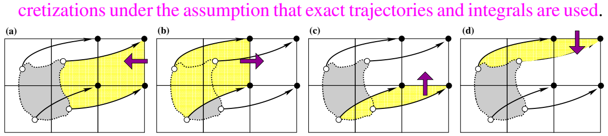
where ρ is the density of the fluid.

2. **Eulerian** type schemes in which the flux of mass through regular (Eulerian) cell walls is tracked (e.g., Lin and Rood 1996). The flux-form discretization is given by

$$\overline{\rho}(t=\Delta t) = \overline{\rho}(t=0) + \frac{1}{\Delta A}\left[\sum_{\ell=1}^{4} m_\ell\right], \qquad (2)$$

where m_ℓ is the total inward mass flux through face ℓ during one Δt. A graphical illustration of m_ℓ is given on the Figures below. For example, the mass flux through the right cell wall is the integral of $\rho(t=0)$ over the yellow shaded area on Fig. a. Similarly for the other cell walls. So by adding up all the yellow areas with the correct signs it is seen that (2) reduces to (1).

- Hence there is an equivalence between the CISL and Eulerian flux-form discretizations under the assumption that exact trajectories and integrals are used.



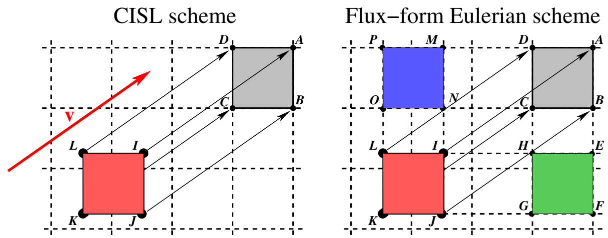
- Here we explicitly consider the cascade CISL scheme of Nair et al. (2002) and the widely used Lin and Rood (1996) scheme given by

$$\overline{\rho}(t=\Delta t) = \overline{\rho}(t=0) + F^x\left[\frac{1}{2}(\overline{\rho}+f^y)\right] + F^y\left[\frac{1}{2}(\overline{\rho}+f^x)\right], \qquad (3)$$

where F^x is the discrete flux divergence in x and f^x is the advective update in x. F and f are referred to as the outer and inner operators, respectively.

Conceptual analysis

Assume a constant (in time and space) wind field. Lagrangian cells moving with the flow will not deform, rotate, expand, or compress but move as solid bodies (see Fig. below).



- In this special case the CISL forecast is simply given by

$$\overline{\rho}(t=\Delta t)\,\Delta A = \int_{(IJKL)}\rho\,dA,$$

where $(IJKL)$ refers to the departure cell with vertices at points A, B, C and D (see Figure above).

Where does the information come from in the CISL forecast? Since CISL schemes explicitly approximate the integral over the departure area the information used for the forecast originates from the departure area as it physically should.

- Where does the information come from in the Lin and Rood (1996) forecast given by (3)? It is not clear in (3) where the information originates but by rewriting (3) in terms of integral operators :

$$\overline{\rho}(t=\Delta t)\,\Delta A = \int_{(IJKL)}\rho\,dA + \frac{1}{2}\left[I^x - \tilde{I}^x + I^y - \tilde{I}^y\right], \qquad (4)$$

where I & $\tilde{I}$ are integral operators associated with the inner and outer operators in (3), respectively,

$$I^x = \int_{(MNOP)}\rho\,dA \quad \text{and} \quad I^y = \int_{(EFGH)}\rho\,dA,$$

(see Fig. above) it does become clear.

If inner and outer operators differ there are spurious non-local contributions to the forecast proportional to the difference between I and $\tilde{I}$ over the non-local areas $(MNOP)$ and $(EFGH)$!

Stability analysis

A linear Von Neumann stability analysis has been performed. The discretized solution is represented as a finite Fourier series and the stability of individual Fourier components is examined.

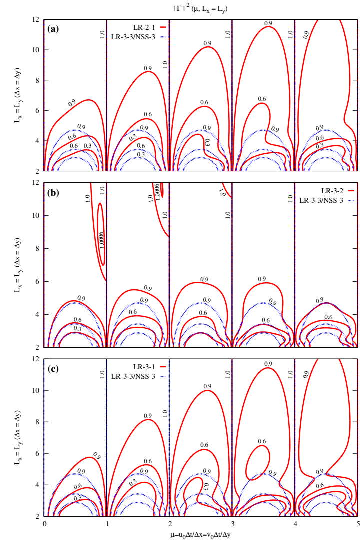


Fig.: *The figures show the squared modulus of the amplification factor for the Lin and Rood (1996, LR) scheme for different combinations of inner and outer operators as a function of symmetric Courant number (μ) and wavelength ($L_x = L_y$). For example, LR-3-2 refers to the LR scheme using a 3rd order outer operator and 2nd order inner operator.*

Note that:

- LR-3-1 and LR-3-2 schemes become increasingly diffusive with increasing Courant number,
- LR-3-2 can be slightly unstable,
- Consistent with the conceptual analysis the LR scheme employing different inner and outer operators have a spurious dependence on μ,
- When the inner and outer operators in the LR scheme are identical this spurious non-local contribution to the forecast disappears and the scheme formally becomes identical to the Nair et al. (2002, NSS) and Zerroukat et al. (2002) schemes. This is, of course, not true for general flows.
- Note that this linear analysis did not include limiters (filters) which are inherently nonlinear: General experience with finite-volume schemes suggests that the unlimited case provides a baseline from which further dissipation and less dispersion is induced by the limiter.

For more information see Lauritzen (2006).

References

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