

On Kelvin Waves in Balance Models

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ABSTRACT

The f -plane linear shallow-water equations support coastal Kelvin waves. These waves propagate along the coast and have zero velocity normal to the coast. It is shown that the balance equations also support coastal Kelvin waves, but these waves differ depending upon the boundary conditions imposed. Three different boundary conditions and resulting Kelvin wave approximations are examined. It is shown that one set of boundary conditions gives balance-model Kelvin waves that are closer to those of the shallow-water equations than the other two boundary conditions.

1. Introduction

The balance equations for a continuously stratified, rotating fluid were first systematically derived by Lorenz (1960). He showed that they are truncations of the primitive equation vorticity and divergence equations through two orders in the usual midlatitude Rossby number scaling and that they have a global energy conservation. Gent and McWilliams (1983) developed consistent boundary conditions for this model in a bounded domain. Allen (1991) proposed an extension to this model that can be written easily as a momentum equation and has a potential vorticity conservation law in addition to energy conservation. He also formulated a different set of boundary conditions than those in Gent and McWilliams (1983). Holm (1996) derived this extended balance model from an asymptotic expansion of Hamilton's principle, demonstrated its Hamiltonian structure, and proposed a companion, but slightly different, balance model in isopycnal coordinates.

To clarify the issue of appropriate boundary conditions for the balance equations, we obtain analytical solutions for midlatitude Kelvin waves in the balance equation approximation and compare them to the linear

shallow-water model Kelvin wave solution. We find that the Kelvin waves resulting from Allen's (1991) boundary conditions, which were also utilized in numerical experiments by Allen et al. (1990), are a better approximation to the shallow-water Kelvin waves than those obtained with the boundary conditions of Gent and McWilliams (1983), which were also employed by Holm (1996). The shallow-water Kelvin wave solution is given in section 2, and the various balance equation approximations to the Kelvin wave are given in section 3. The conclusions are briefly stated in section 4.

2. Linear shallow-water equations

We consider the linearized shallow-water equations in a fluid of average depth H and a free surface elevation of η . The equations are

$$\mathbf{u}_t + f\mathbf{k} \times \mathbf{u} + g\nabla\eta = 0, \quad (2.1)$$

$$\eta_t + H\nabla \cdot \mathbf{u} = 0, \quad (2.2)$$

where \mathbf{u} is the horizontal velocity, \mathbf{k} is a unit vertical vector, f is the constant Coriolis frequency, g is the gravitational constant, t is time, and subscript t denotes partial differentiation. These equations imply conservation of a linearized potential vorticity, in the form

$$[H\mathbf{k} \cdot \nabla \times \mathbf{u} - f\eta]_t = 0. \quad (2.3)$$

The energy equation is

$$E_t + g\nabla \cdot (\eta\mathbf{u}) = 0, \quad (2.4a)$$

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where

$$E = \frac{1}{2}[\mathbf{u} \cdot \mathbf{u} + (g/H)\eta^2]. \quad (2.4b)$$

Thus, in bounded domains the boundary condition of no normal flow at the domain boundary ensures global energy conservation, that is, conservation of the area integral of E .

We now consider an eastern ocean boundary located at $x = 0$. The f -plane equations support coastal Kelvin waves that travel northward or southward in the Northern and Southern Hemispheres, respectively. The Northern Hemisphere Kelvin wave solution is given by

$$u = 0, \quad (2.5a)$$

$$(v, \eta) = (c, H) \exp[fx/c + i(ly - \omega t)] \quad (2.5b,c)$$

$$\omega = cl, \quad (2.6)$$

where $c = (gH)^{1/2}$ is the gravity wave speed and where the dispersion relation (2.6) implies that the waves are nondispersive. The zonal decay scale of the waves away from the boundary is equal to c/f .

3. Linear balance equations

The linear balance equations are an approximation to the shallow-water equations that retain the time derivative of the rotational component of the velocity only. They take the form

$$\mathbf{u}_{Rt} + f\mathbf{k} \times \mathbf{u} + g\nabla\eta = 0, \quad (3.1)$$

where

$$\mathbf{u} = \mathbf{u}_R + \mathbf{u}_D = \mathbf{k} \times \nabla\psi + \nabla\chi, \quad (3.2)$$

in addition to the exact linear height equation (2.2). Equations for the vertical component of vorticity and the horizontal divergence are typically considered in place of the momentum equations (3.1). In terms of the variables (ψ, χ, η) , these are

$$\nabla^2\psi_t + f\nabla^2\chi = 0, \quad (3.3a)$$

$$f\nabla^2\psi = g\nabla^2\eta, \quad (3.3b)$$

while (2.2) is

$$\eta_t + H\nabla^2\chi = 0. \quad (3.4)$$

No time derivative appears in the divergence equation (3.3b) and this filters out high-frequency gravity-inertial waves. A relevant point, evident from the governing equations (3.3a,b) and (3.4), is that arbitrary solutions of $\nabla^2\psi = 0$ and $\nabla^2\chi = 0$ may be added to ψ and χ , respectively, without affecting the satisfaction of (3.3a,b) and (3.4). Proper boundary conditions are necessary to ensure appropriate solutions for ψ and χ .

The linear balance equations conserve a linearized potential vorticity,

$$[H\mathbf{k} \cdot \nabla \times \mathbf{u}_R - f\eta]_t = 0, \quad (3.5)$$

which, since $\mathbf{k} \cdot \nabla \times \mathbf{u} = \mathbf{k} \cdot \nabla \times \mathbf{u}_R$, is identical to the shallow-water equation (2.3). The energy equation is

$$E_{Rt} + g\nabla \cdot (\eta\mathbf{u}) + \nabla \cdot (\chi\mathbf{u}_{Rt}) = 0, \quad (3.6a)$$

where

$$E_R = \frac{1}{2}[\mathbf{u}_R \cdot \mathbf{u}_R + (g/H)\eta^2]. \quad (3.6b)$$

In bounded domains, the area integral of E_R will be conserved if, at the boundary, there is no normal flow

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad (3.7)$$

and if

$$\oint_C \chi\mathbf{u}_{Rt} \cdot \mathbf{n} ds = 0, \quad (3.8)$$

where \mathbf{n} is the outward pointing unit vector normal to the boundary, and the integral in (3.8) is around the boundary contour C . Thus, global conservation of the energy E_R cannot be assured merely by the boundary condition of no normal flow at the domain boundary and a second boundary condition is needed. This is standard for the balance equations because the specification of unique solutions for both ψ and χ requires an extra boundary condition in addition to (3.7) (Gent and McWilliams 1983; Allen et al. 1990).

Global energy conservation in Eq. (3.6) can be assured by at least three different choices of boundary conditions. They are

(A) Allen (1991)

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad (3.9a)$$

$$(g\nabla\eta + f\mathbf{k} \times \mathbf{u}_R) \cdot \mathbf{n} = 0, \quad (3.9b)$$

(B) Gent and McWilliams (1983) and Holm (1996)

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{u}_R \cdot \mathbf{n} = 0, \quad (3.10a,b)$$

(which implies $\mathbf{u}_D \cdot \mathbf{n} = 0$), and

(C)

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \chi = 0. \quad (3.11a,b)$$

Note that the Allen (1991) conditions (A) ensure conservation of energy E_R since (3.1) and (3.9b) imply that, on the boundary,

$$\mathbf{u}_{Rt} \cdot \mathbf{n} + f(\mathbf{k} \times \nabla\chi) \cdot \mathbf{n} = 0, \quad (3.12)$$

so that (3.8) is satisfied because the integrand is a perfect differential.

With boundary conditions (A), the Kelvin wave solution at an eastern ocean boundary is

$$u = 0, \quad (3.13a)$$

$$(v, \eta) = (f/k, H) \exp[kx + i(ly - \omega t)], \quad (3.13b,c)$$

$$\omega = fl/k, \quad (3.14)$$

where

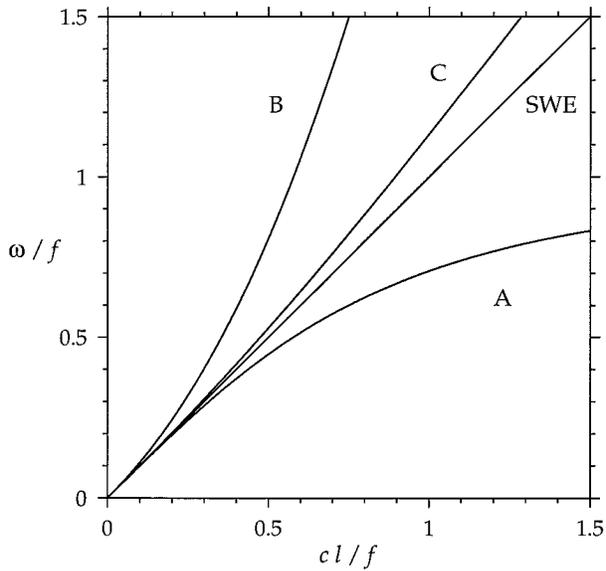


FIG. 1. Dispersion relations for the linear shallow-water equations SWE (2.6) and for the linear balance equations with boundary conditions A (3.14), B (3.18), and C (3.23b).

$$k = (l^2 + f^2/c^2)^{1/2}, \tag{3.15}$$

and where, for definiteness in what follows, we assume $l \geq 0$. In comparison to the shallow-water equation (2.5), this Kelvin wave solution also has zero zonal velocity and a single decay scale away from the coast equal to $1/k$. The dispersion relation (3.14) is plotted in Fig. 1 along with the shallow-water relation (2.6). In the small wavenumber low-frequency limit, the A dispersion relation (3.14) gives

$$\omega = cl \left(1 - \frac{1}{2} c^2 l^2 / f^2 + \dots \right) \text{ for } l \ll f/c, \tag{3.16}$$

which is second-order accurate in this important limit. Likewise, the error in the decay scale $1/k$ and in the amplitude of v is small $O[(cl/f)^2]$ for $l \ll f/c$. As shown in Fig. 1, $\omega/f \sim 1$ for $l \gg f/c$ so the frequency is finite as the meridional wavenumber becomes very large.

Boundary conditions B impose that the normal components of both the rotational and divergent velocities are zero. They are discussed in Gent and McWilliams (1983), where they are labeled choice 1, and by Holm (1996) in his derivation of the balance equations based on Hamiltonian dynamics. The choice 2 boundary conditions discussed in Gent and McWilliams (1983) are equivalent to choice 1 in this linear f -plane model. For a coastal Kelvin wave, boundary conditions B imply that the solution cannot be written purely in terms of the spatial function $\exp(kx)$. Additional components must be added to ψ and χ that satisfy $\nabla^2 \psi = 0$ and $\nabla^2 \chi = 0$, respectively, and so have an $\exp(lx)$ spatial dependence. The eastern boundary Kelvin wave solution that satisfies boundary condition B is given by

$$u = i \frac{c^2 \omega l}{f^2} \exp[i(ly - \omega t)] \times [\exp(kx) - \exp(lx)], \tag{3.17a}$$

$$v = \frac{c^2}{f^2} \exp[i(ly - \omega t)] \times [(fk - \omega l) \exp(kx) + \omega l \exp(lx)], \tag{3.17b}$$

$$\eta = H \exp[kx + i(ly - \omega t)], \tag{3.17c}$$

$$\omega = fl/(k - l). \tag{3.18}$$

The dispersion relation in (3.18) is plotted in Fig. 1 and has two undesirable features compared to the dispersion relation (3.14) for boundary conditions A. The first is that in the small wavenumber, low-frequency limit the frequency is only first-order accurate, that is,

$$\omega = cl(1 + cl/f + \dots) \text{ for } l \ll f/c, \tag{3.19}$$

and the second is that the frequency becomes very large as the meridional wavenumber gets large; that is, $\omega \sim 2(cl)^2/f$ for $l \gg f/c$. In addition, compared to the shallow-water solution (2.5), this solution has zero zonal velocity only at the boundary and has two decay scales away from the coast equal to $1/k$ and $1/l$. In the small meridional wavenumber limit $l \ll f/c$, however, the relative magnitude of the component of v with decay scale $1/l$ is small $O[(cl/f)^2]$, as is the magnitude of the nonzero zonal velocity. It is clear, nevertheless, that in the limit $l \ll f/c$, that is, for $\omega \ll f$, the coastal Kelvin wave in the balance equations with boundary conditions A is a better approximation to the shallow-water Kelvin wave than that with boundary conditions B.

Boundary conditions C impose that the normal component of velocity and the divergent potential χ are zero on the boundary. The eastern boundary Kelvin wave solution that satisfies boundary conditions C is given by

$$u = \frac{ic^2}{f^2} (\omega k - fl) \exp[i(ly - \omega t)] \times [\exp(kx) - \exp(lx)], \tag{3.20a}$$

$$v = \frac{c^2}{f^2} \exp[i(ly - \omega t)] \times [(fk - \omega l) \exp(kx) + (\omega k - fl) \exp(lx)], \tag{3.20b}$$

$$\eta = H \exp[kx + i(ly - \omega t)], \tag{3.20c}$$

$$\omega(f + \omega) = f^2 l / (k - l). \tag{3.21}$$

The dispersion relation (3.21) has two roots

$$\omega_{\pm} = -\frac{1}{2} f [1 \pm (1 + 4l/(k - l))^{1/2}], \tag{3.22}$$

where, for $l \ll f/c$,

$$\omega_+ = -f(1 + cl/f + \dots), \quad (3.23a)$$

$$\omega_- = cl \left(1 + \frac{1}{2} c^2 l^2 / f^2 + \dots \right). \quad (3.23b)$$

Thus, the solution with ω_- corresponds closely to a Kelvin wave while the solution with ω_+ is a spurious mode of oscillation, which is an unfavorable aspect of boundary conditions C. The dispersion relation (3.23b) for ω_- is plotted in Fig. 1. Compared to the shallow-water solution (2.5), the ω_- solution shares the disadvantages of the B boundary condition solution in having the zonal velocity zero only at the boundary and having two decay scales away from the boundary. However, in the limit $l \ll f/c$, the nonzero zonal velocity in the ω_- solution is small $O[(cl/f)^3]$ as is the relative magnitude of the component of v with decay scale $1/l$. Finally, note that the frequency becomes large as the meridional wavenumber gets large; that is, $\omega_- \sim \sqrt{2}cl$ for $l \gg f/c$.

4. Conclusions

The midlatitude coastal Kelvin waves are well represented in linearized balance models for $l \ll f/c$, or equivalently $\omega \ll f$, using the boundary conditions formulated by Allen (1991) when he proposed the balance equations based on momentum. The Kelvin waves are

not as well represented in this limit using other boundary conditions including the choice 1 of Gent and McWilliams (1983), which sets the normal components of both the rotational and divergent velocities to zero.

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