6 Two-layer shallow water theory.

We will now go on to look at a shallow water system that has two layers of different density. This is the next level of complexity and a simple starting point for understanding the behaviour of vertically stratified flow. It is also the simplest model within which baroclinic instability can be understood - the process the generates synoptic scale weather systems (this will be in section 8).

The set up of the system to be considered is shown in Fig. 1. It has bottom topography of height h_B, a lower layer of density ρ_2 and an upper layer of density ρ_1 where $\rho_1 < \rho_2$. Above this it is assumed that there is a fluid of negligible density so the pressure can be taken to be zero at the surface. The depth of the individual layers 1 and 2 are h_1 and h_2 respectively. The depth of the fluid in each layer is much shallower than the horizontal scale of the layer and so it is assumed that hydrostatic balance applies and the pressure at any point may be found by integrating the density times gravity downward. The pressure is assumed to vary continuously across the interface between the two layers but the density does not. It jumps discontinuously from ρ_2 to ρ_1. There can therefore be a discontinuity in the velocity field at the interface. Considering a geostrophically balanced flow

$$f \times \vec{v}_H = -\frac{1}{\rho}\nabla p$$

At the interface the pressure varies continuously so considering an infinitesimal region straddling the interface the horizontal pressure gradient will effectively be the same in layer 1 and layer 2. But, the density is not, so a discontinuity in the velocity field results.

6.1 Momentum balance in the two-layer shallow water system

The shallow water momentum balance equation is a

$$\frac{D\vec{v}_H}{Dt} + f \times \vec{v}_H = -\frac{1}{\rho}\nabla p \tag{1}$$

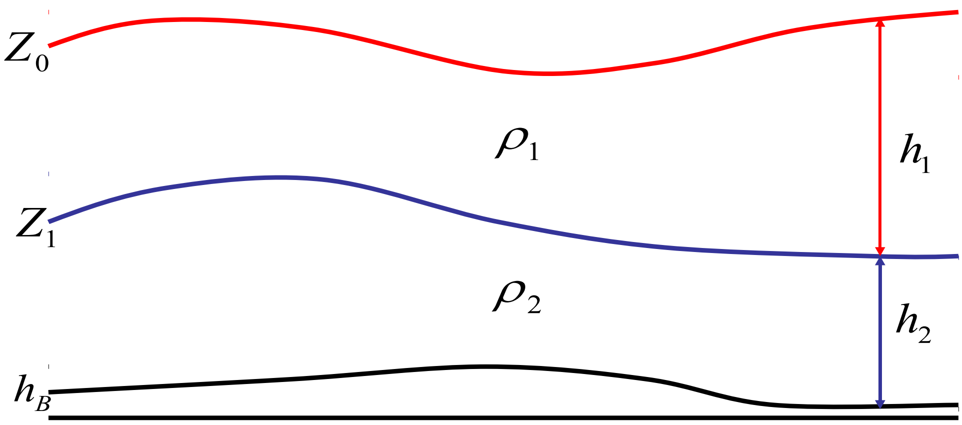

Figure 1: Schematic of the two-layer shallow water system

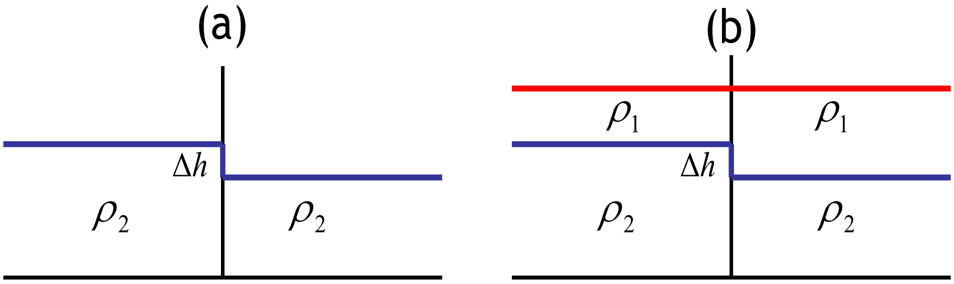

Figure 2: Schematic illustration of two initially unbalanced situations (a) for a single layer of fluid of density ρ_2 and (b) for a two-layer situation with a fluid of density ρ_1 lying on top of a fluid of density ρ_2

In the single layer case the pressure gradient was simply gravity time the gradient in the free surface height. The same quation also applies to the two layer case but expressions for the pressure gradients must be determined.

- In Layer 1: At a height z in the first layer the pressure will be given by

$$p_1 = \rho_1 g(Z_o - z) \tag{2}$$

- In layer 2: At a height z in the second layer the pressure will be given by

$$\begin{aligned} p_2 &= \rho_1 g(Z_o - Z_1) + \rho_2 g(Z_1 - z) \\ &= \rho_1 g Z_o - \rho_1 g Z_1 + \rho_2 g Z_1 - \rho_2 g z \\ &= \rho_1 g Z_o + \rho_2 g' Z_1 - \rho_2 g z \end{aligned} \tag{3}$$

  where $g' = \frac{\rho_2 - \rho_1}{\rho_2} g$ is the **reduced gravity**. The meaning of this reduced gravity can be understood by considering the adjustment of a fluid that's initially unbalanced as shown in Fig, 2. In the case of a single layer (a) the pressure gradient between the two sides of the interface is given by $\Delta p = \rho_2 gh$. In the second case (b) where there is another fluid of density ρ_1 on top of the original layer the pressure difference between the two sides of the interface is now $\Delta p = (\rho_2 - \rho_1)gh$. This can be re-written as $\Delta p = \rho_2(\frac{\rho_2 - \rho_1}{\rho_2})gh = \rho_2 g' h$. It can therefore be seen that the difference in pressure gradient between the two cases is effectively the same as if you reduced the gravitational constand by a factor $(\rho_2 - \rho_1)/\rho_2$. The buoyancy effects associated with the second fluid above have the effect of reducing the effects associated with the gravitational restoring force and so the fluid will adjust in the same way as if the gravity was reduced to the reduced gravity.

So, back to the pressure gradients in 2 and 3 and inputting these into 1 gives, for the first layer,

$$\frac{D\vec{v}_1}{Dt} + (f \times \vec{v}_1) = -g\nabla Z_o \tag{4}$$

This is the same as the momentum balance for the single layer case, the pressure gradient is only associated with the variations in the free surface height. For the second layer momentum balance becomes

$$\begin{aligned}\frac{D\vec{v}_2}{Dt} + (f \times \vec{v}_2) &= -\frac{1}{\rho_2}(\rho_1 g \nabla Z_o - \rho_2 g \nabla Z_1) \\ &= -\frac{\rho_1}{\rho_2} g \nabla Z_o - \frac{\rho_2}{\rho_2} g' \nabla Z_1 \\ &\sim -g \nabla Z_o - g' \nabla Z1\end{aligned} \tag{5}$$

In this last step we have made use of the **Boussinesq Approximation**. Under this approximation you can neglect the difference in density between the two layers except where it is being multiplied by gravity i.e. you assume that the differences in inertia of the two fluids is negligible but their weights are different. What this means is that we can assume $\rho_1/\rho_2 \sim 1$ but we cannot neglect the difference between the densities in the reduced gravity. If we neglected the difference in the reduced gravity we would just be back to the usual single layer equations.

Eq. 5 demonstrates that in layer 2 the pressure gradient is related both to the variations in the free surface height and the slope of the interface between the two fluids. Considering the steady state geostrophically balanced momentum equations

$$f \times \vec{v}_1 = -g \nabla Z_o$$

$$f \times \vec{v}_2 = -g \nabla Z_o - g' \nabla Z_1$$

Taking the difference between these two equations gives

$$f \times (\vec{v}_1 - \vec{v}_2) = g' \nabla Z_1$$

This is the thermal wind balance equation for the two layer shallow water system. It relates the vertical wind shear across the interface to the slope of the interface i.e. it relates the vertical wind shear to horizontal density gradients. The significant difference between the two layer shallow water system and the single layer shallow water system is that there are now variations with height - there is **Baroclinicity**.

6.2 The PV equation for the two layer system

Starting from the momentum balance equations

$$\frac{D\vec{v}_1}{Dt} + f \times \vec{v}_1 = -g \nabla Z_o \tag{6}$$

$$\frac{D\vec{v}_2}{Dt} + f \times \vec{v}_2 = -g \nabla Z_o - g' \nabla Z_1 \tag{7}$$

The process of obtaining the PV conservation equation from the momentum balance equations for the single layer is discussed in Section 4.1.3. This begins with taking the curl of momentum balance to obtain the vorticity equation. In layer 1, Eq. 6 is exactly the same as the shallow water momentum balance for a single layer so we just obtain the same vorticity equation as in that case. For layer 2 the RHS of momentum balance differs as there is an additional term $-g' \nabla Z_1$. However, since the vorticity equation is formed by taking the curl of momentum balance this doesn't actually matter as the curl of a divergence is zero. So the vorticity equation is the same for each layer and is given by

$$\frac{\partial \zeta_i}{\partial t} + \vec{v_i}.\nabla(\zeta_i + f) = -(\zeta_i + f)\nabla.\vec{v_i} \tag{8}$$

where the subscript i represents whichever layer (1 or 2) is being considered. The potential vorticity equation is then formed by making use of mass conservation

$$\frac{Dh_i}{Dt} + h_i \nabla.\vec{v_i} = 0, \tag{9}$$

where h_i is the thickness of the layer to re-write the divergence of the velocity on the right hand side of 8 as follows

$$\nabla.\vec{v_i} = -\frac{1}{h_i}\frac{Dh_i}{Dt}$$

This gives,

$$\frac{\partial(\vec{\zeta_i} + f)}{\partial t} + \vec{v_i}.\nabla(\vec{\zeta} + f) = \frac{\vec{\zeta} + f}{h_i}\frac{Dh_i}{Dt}$$

The left hand side may be written as the material derivative of the absolute vorticity giving

$$\frac{D(\vec{\zeta} + f)}{Dt} = \frac{\vec{\zeta} + f}{h_i}\frac{Dh_i}{Dt}$$

which, making use of the product rule, can be written as

$$h_i^2 \frac{D}{Dt}\left(\frac{\vec{\zeta_i} + f}{h_i}\right) = 0.$$

Therefore

$$\frac{Dq_i}{Dt} = 0 \qquad q_i = \frac{\vec{\zeta_i} + f}{h_i} \tag{10}$$

This is potential vorticity conservation for the two layer case. This can be seen to be identical to PV conservation for the single layer case (Eq. 6 in section 4.1.3).

6.3 Q-G scaling for the two layer shallow water system

In deriving the Q-G form of the two-layer shallow water PV equation, similar assumptions to those in Section 5.1 have to be made i.e.

- The Rossby number is small $\vec{v_a}/\vec{v}_g \sim O(R_o)$
- Variations in the thickness of each layer are small. The thickness of each layer is given by $h_i = H_i + h'_i$, H_i is the mean layer thickness and h'_i is a perturbation to that thickness. We assume the perturbation is small compared to the mean layer depth such that $|h'_i|/H_i \sim O(R_o)$
- Variations of the coriolis parameter with latitude are small $\beta y/f_o \sim R_o$.
- The timescale to be considered is advective and the advection is done by the geostrophic velocity $D/Dt = \partial/\partial t + u_g\partial/\partial x + v_g\partial/\partial y$

Starting with PV of the form

$$q_i = \frac{\vec{\zeta}_i + f}{H_i + h'_i} = \frac{\vec{\zeta}_i + f}{H_i}\left(1 + \frac{h'_i}{H_i}\right)^{-1}$$

an making use of the fact that $h_i/H_i << 1$ this can be taylor expanded to give

$$q_i = \frac{1}{H_i}\left(\vec{\zeta}_i + f - \frac{\vec{\zeta}_i h'_i}{H_i} - f\frac{h'_i}{H_i}\right)$$

A scale analysis on this, retaining only the term of order R_o, gives the QG PV equation for each layer as

$$\frac{Dq_i}{Dt} \qquad q_i = \vec{\zeta}_{gi} + \beta y - f_o\frac{h'_i}{H_i} \tag{11}$$

This is analogous to the single layer Q-G PV equation but here the thickness of the layer can be changed from the top or from the bottom. This is an equation for the time rate of change of the geostrophic vorticity but it's still in terms of the free surface height perturbation (h'_i). We need an equation that describes the time evolution of the geostrophic stream function solely in terms of the geostrophic quantities themselves. We can make use of geostrophic balance to obtain expressions for the thickness variations h'_i in terms of the geostrophic stream function.

Geostrophic balance gives

$$f_o \times \vec{v_1} = -g\nabla Z_o$$

$$f_o \times \vec{v_2} = -g\nabla Z_o - g'\nabla Z_1$$

Now, Z_o and Z_1 are related to the layer thicknesses and bottom topography by

$$Z_o = h_1 + h_2 + h_B \qquad\qquad Z_1 = h_2 + h_B$$

Writing each layer depth in terms of a mean layer depth and a perturbation ($h_2 = H_2 + h'_2$ and $h_1 = H_1 + h'_1$) the geostrophic balance equations can be written as

$$f_o \times \vec{v_1} = -g\nabla(h'_1 + h'_2 + h_B)$$

$$f_o \times \vec{v_2} = -g\nabla(h'_1 + h'_2 + h'_B) - g'\nabla(h'_2 + h_B)$$

Since the geostrophic stream function is related to the geostrophic velocities by

$$u_g = -\frac{\partial\psi}{\partial y} \qquad\qquad v_g = \frac{\partial\psi}{\partial x}$$

it can be seen that the geostrophic stream functions in each layer can be written as

$$\psi'_1 = \frac{g}{f_o}(h'_1 + h'_2 + h_B) \tag{12}$$

$$\psi'_2 = \frac{g}{f_o}(h'_1 + h'_2 + h_B) + \frac{g'}{f_o}(h'_2 + h_B) \tag{13}$$

Some rearranging gives expressions for the layer depths in terms of the geostrophic stream functions

$$h'_2 = \frac{f_o}{g'}(\psi_2 - \psi_1) - h_B \tag{14}$$

$$h'_1 = \frac{f_o}{g}\psi_1 + \frac{f_o}{g'}(\psi_1 - \psi_2) \tag{15}$$

Given that we are considering the Boussinesq approximation where the difference in the density between the two layers is small then that is equivalent to stating that $g'/g << 1$. Therefore the first term in the equation for h'_1 may be neglected compared to the second term. Making this approximation is actually equivalent to stating that the lid is rigid because it implies that $h'_1 = -h'_2 - h_B$ i.e. the variations in the thickness of layer 1 come only from the variations in the bottom topography or the variations in the thickness of layer two. Alternatively $h'_1 + h'_2 + h'_B = 0$ i.e. there are no variations in the total height of the free surface height.

Using 14 and 15 provides us with the Q-G version of the potential vorticity solely in terms of the geostrophic stream function.

$$\boxed{\frac{Dq_1}{Dt} \qquad q_1 = \vec{\zeta_1} + \beta y - \frac{f_o^2}{gH}(\psi_1 - \psi_2)} \tag{16}$$

$$\boxed{\frac{Dq_2}{Dt} \qquad q_2 = \vec{\zeta_2} + \beta y + \frac{f_o^2}{gH}(\psi_1 - \psi_2) + \frac{f_o}{H_2}h_B} \tag{17}$$

where $\vec{\zeta_i}$ and $\vec{\zeta_2}$ can be written as $\nabla^2\psi_1$ and $\nabla^2\psi_2$. This can be compared with the single layer PV

$$q = \vec{\zeta} + \beta y - \frac{f_o^2}{gH}\psi \qquad\qquad \psi = \frac{g}{f_o}h' \tag{18}$$

In the single layer case in order to conserve potential vorticity the absolute vorticity would adjust to changes in the free surface height. In the two layer case the absolute vorticity will adjust to changes in the thickness of both layers 1 and 2 and in opposite senses e.g. say there was an increase in h'_2. From the difference between 13 and 12 this corresponds to an increase in $(\psi'_2 - \psi'_1)$. Since the appearance of this term in q_1 and q_2 differs by a minus sign if can be seen that this will have opposite effects on the relative vorticity in each layer. In layer 2 the relative vorticity will have to decrease to conserve PV whereas in the upper layer the relative vorticity will have to increase.

6.4 Rossby waves in the two layer shallow water system

Starting from the PV conservation equations 16 and 17. For simplicity consider the case where $H_1 = H_2 = H$ and the background state is a fluid at rest. The stream functions ψ_1 and ψ_2 therefore only consist of a perturbation part ψ'_1 and ψ'_2 and linearising (i.e. neglecting the terms of order primed quantities squared the material derivative becomes simply the time derivative (since the velocity field only consists of perturbation velocities (u' and v').

Therefore in layer 1 PV conservation becomes

$$\frac{\partial}{\partial t}\left[\nabla^2\psi'_1 - \frac{f_o^2}{g'H}(\psi'_1 - \psi'_2)\right] + \beta\frac{\partial\psi'_1}{\partial x} = 0 \tag{19}$$

where the term $\frac{\partial}{\partial t}\beta y$ has been re-written as $\beta v = \beta\frac{\partial\psi'_1}{\partial x}$. Similarly in layer 2 the PV equation becomes

$$\frac{\partial}{\partial t}\left[\nabla^2\psi'_2 + \frac{f_o^2}{g'H}(\psi'_1 - \psi'_2)\right] + \beta\frac{\partial\psi'_2}{\partial x} = 0 \tag{20}$$

These are two coupled equations i.e. the stream functions ψ'_1 and ψ'_2 have to satisfy both of them. However, they can be transformed into two uncoupled equations by adding 19 and 20 and subtracting 20 from 19.

This gives

$$\frac{\partial}{\partial t}\nabla^2(\psi'_1 + \psi'_2) + \beta\frac{\partial}{\partial x}(\psi'_1 + \psi'_2) = 0 \tag{21}$$

and

$$\frac{\partial}{\partial t}\nabla^2(\psi'_1 - \psi'_2) - \frac{2f_o^2}{g'H}(\psi_1 - \psi_2) + \beta\frac{\partial}{\partial x}(\psi'_1 - \psi'_2) = 0 \tag{22}$$

These are two separate equations: one for the sum of the stream functions and one for their difference. We can now define the **Barotropic** component of the stream function to be the average stream function of the two layers i.e.

$$\hat{\psi} = \frac{\psi'_1 + \psi'_2}{2}.$$

The **Baroclinic** part of the stream function is defined to be the difference between the stream functions of the layers i.e. the part that varies with height

$$\tilde{\psi} = \frac{\psi'_1 - \psi'_2}{2}.$$

The stream function in each layer therefore consists of the sum of the barotropic and the baroclinic component. Equations 21 and 22 can therefore be written as

$$\frac{\partial}{\partial t}\left[\nabla^2\hat{\psi}\right] + \beta\frac{\partial\hat{\psi}}{\partial x} = 0 \tag{23}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\tilde{\psi} - \frac{2}{g'}\frac{f_o^2}{H}\tilde{\psi} + \beta\frac{\partial\tilde{\psi}}{\partial x}\right] = 0 \tag{24}$$

These can then be solved by assuming a wave solution to find the barotropic and baroclinic parts of the stream function. For the barotropic component (23) the equation is exactly the same as the equation that was used to examine Rossby waves in the single layer shallow water system (See Section 4.2.4). The solutions are therefore the same as for the single layer case i.e. Rossby waves that propagate to the west and obey the following dispersion relation

$$\hat{\omega} = \frac{-\beta k}{(k^2 + l^2)} \tag{25}$$

The baroclinic mode has an extra term but plugging in a wave solution of the form $\tilde{\psi} = \psi_o exp(i(kx + ly - \omega t))$ it can be demonstrated that the baroclinic stream function is also a Rossby wave propagating to the west but it obeys the following dispersion relation.

$$\tilde{\omega} = \frac{-\beta k}{(k^2 + l^2 + \frac{2}{L_D^2})} \tag{26}$$

where L_D here is the internal Rossby radius of deformation ($\sqrt{g'H}/f_o$). Comparing the phase speeds (ω/k) of these two solutions it can be seen that the baroclinic mode phase speed is slower than the barotropic mode.

To conclude, in the two layer system there are two solutions that satisfy the Q-G potential vorticity conservation. A barotropic mode that consists of a stream function perturbation that is constant with height i.e. it does not involve any discontinuities at the interface or any slope of the interface, and a baroclinic mode i.e. a mode that varies with height and will therefore involve a slope of the interface between the two layers.