Problem Sheet 2: Due Thurs 17th Feb

1. Vorticity and Circulation Examples

(a) Demonstrate that the vertical component of the Earths vorticity relative to a tangent plane on the Earths surface at a latitude ϕ is given by $2\Omega sin\phi$, where Ω is the angular velocity of the Earth. (*Hint: Consider the Earth to be an object in solid body rotation and use cylindrical coordinates*)

(b) Consider a horizontal circulation around the origin where the tangential velocity is given by $v_T = K/r$ where K is a constant. Prove that the vertical component of vorticity is zero for $r > 0$. What happens at $r = 0$?

(c) A cylindrical column of air at $50^o N$ of radius 200km initially has zero tangential velocity at its perimeter. Using Kelvins circulation theorem, obtain the tangential velocity around the perimeter of the loop if it expands to twice its original radius. You may consider the column to be on an f-plane. What other assumptions have you made by using Kelvins circulation theorem?

2. Potential vorticity in the presence of friction

Potential vorticity conservation in the absence of friction or diabatic heating was derived in class by considering a fluid element that is bounded by surfaces of a materially conserved quantity. Work through these steps again but now include a frictional force per unit mass ($\vec{F_f}$) and demonstrate that in the presence of friction the PV equation is given by

$$\frac{Dq}{Dt} = \frac{\nabla\theta}{\rho}.(\nabla \times \vec{F_f}), \qquad q = \frac{\vec{\omega_a}.\nabla\theta}{\rho} \tag{1}$$

Give an interpretation of this.

3. Potential vorticity in the shallow water model

In the shallow water system, on an f-plane, the absolute and potential vorticity associated with horizontal motion are given by

$$\omega_a = f_o + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad q = \frac{\omega_a}{h} \tag{2}$$

where we are ignoring the horizontal component of the Earth's rotation. The bottom surface is taken to be flat and h is the height of the free surface given by $h = h_o(1 + \eta)$ where h_o is the mean free surface height and η is a free surface height perturbation.

Consider a flow of small rossby number. Demonstrate that in the limit $\eta << 1$ and linearizing in η that the potential vorticity may be written as

$$q \simeq q_o(1 - \eta + L_D^2 \nabla^2 \eta) \tag{3}$$

where $L_D = \sqrt{gh_o}/f$ (known as the Rossby radius of deformation) and $q_o = f/h_o$ is the planetary potential vorticity.

(b) We may now define a potential vorticity perturbation $q' = q - q_o$ such that

$$\frac{q'}{q_o} = L_D^2 \nabla^2 \eta - \eta \tag{4}$$

Consider a potential vorticity perturbation that is periodic in the x direction

$$q' = \epsilon q_o sin(kx) \tag{5}$$

where ϵ and k are constants. Solve equation 4 for the geostrophic velocities and sketch the fluid height and geostrophic velocity as a function of x.

(c) For the more general situation of a potential vorticity perturbation that is a function of both x and y, describe how you could make use of 4 together with geostrophic balance and potential vorticity conservation to work out, numerically, the geostrophic wind as a function of time. (assume you can solve 4 for η. No detailed calculations are necessary, just provide a short description of the procedure you would use.)

4. Potential vorticity conservation example

An air column at 45^oN with zero relative vorticity stretches from the surface to the tropopause which we assume can be considered to be a rigid lid at 10km. The air column moves zonally over the Rocky mountain plateau to a height of 3km. What is the relative vorticity of the column at this new position? It now moves South to 30^oN. What is the new relative vorticity? (Assume that the density does not change).

5. Considering the zonal mean temperature structure of the atmosphere, provide a qualitative explanation for the prevalence of cyclones in the mid-latitudes.